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On the Translated Whitney Numbers and their Combinatorial Properties

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Authors' contributions

This work was carried out in collaboration between both authors. Author MMM designed the study and managed the literature searches. Author AMD wrote the first draft of the manuscript with some modifications to the original idea of the study. Author MMM proof read and carried out the revision of the manuscript. Both authors read and approved the final manuscript.

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ABSTRACT

In this paper, we further develop the study of the translated Whitney numbers by deriving more combinatorial properties such as more recurrence relations, exponential and rational generating functions and the orthogonality and inverse relations. To achieve this goal, we introduce the "signed" translated Whitney numbers of the first kind. A relationship between these numbers and the Bernoulli polynomials is also briefly discussed.

Keywords: Translated whitney numbers; whitney numbers; stirling numbers.

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1. INTRODUCTION

The *Whitney numbers of the first and second kind of Dowling lattices* $Q_n(G)$, denoted by $w_m(n,k)$ and $W_m(n,k)$, respectively, were defined by Benoumhani [1] in terms of the generating functions

$$m^n(x)_n = \sum_{k=0}^{n} w_m(n,k)(mx+1)^k \qquad (0.1)$$

and

$$(mx+1)^n = \sum_{k=0}^{n} m^k W_m(n,k)(x)_k, \qquad (0.2)$$

where $(x)_n = x(x-1)\cdots(x-n+1)$ is the n-th order falling factorial of x. Fundamental properties of these numbers were already established by Benoumhani in [1] and [2]. The numbers $w_m(n,k)$ and $W_m(n,k)$ are actually related to the well-known *Stirling numbers of the first kind* $s(n,k)$ *and second kind* $S(n,k)$ as follow

$$w_1(n,k) = s(n+1,k+1)$$

and

$$W_1(n,k) = S(n+1,k+1).$$

One may see Comtet [3] for a more detailed discussion on some of the properties of the numbers $s(n,k)$ and $S(n,k)$. Some generalizations of these numbers were already considered in [4] and [5]. Another generalizations are the *translated Whitney numbers of the first kind* $\widetilde{w}_{(\alpha)}(n,k)$ *and second kind* $\widetilde{W}_{(\alpha)}(n,k)$ which were introduced by Belbachir and Bousbaa [6]. $\widetilde{w}_{(\alpha)}(n,k)$ and $\widetilde{W}_{(\alpha)}(n,k)$ count the number of permutations of n elements with k cycles such that the element of each cycle can mutate in α ways, except the dominant one and the partitions of the set $\{1,2,3,\ldots,n\}$ into k subsets such that each element of each subset can mutate in α ways, except the dominant one, respectively. One may see [6] for a detailed discussion of these combinatorial interpretations. Also, several properties for $\widetilde{w}_{(\alpha)}(n,k)$ and $\widetilde{W}_{(\alpha)}(n,k)$ were already presented in [6] as particular cases of the properties obtained by Hsu and Shiue [7] for the unified generalized Stirling numbers $S(n,k;\alpha,\beta,\gamma)$ and $S(n,k;\beta,\alpha,-\gamma)$. To mention a few of these properties, we have the *triangular recurrence relations*

$$\widetilde{w}_{(\alpha)}(n+1,k+1) = \widetilde{w}_{(\alpha)}(n,k) + (\alpha n)\cdot \cdot\widetilde{w}_{(\alpha)}(n,k+1), \qquad (0.3)$$

$$\widetilde{W}_{(\alpha)}(n,k) = \widetilde{W}_{(\alpha)}(n-1,k-1) + \alpha k\widetilde{W}_{(\alpha)}(n-1,k); \qquad (0.4)$$

and the *horizontal generating functions*

$$(x|\alpha)_n = \sum_{k=0}^{n} \widetilde{w}_{(\alpha)}(n,k)x^k, \qquad (0.5)$$

$$x^n = \sum_{k=0}^{n} \widetilde{W}_{(\alpha)}(n,k)(x|-\alpha)_k, \qquad (0.6)$$

where

$$(x|\alpha)_n = \prod_{i=0}^{n-1}(x+i\alpha).$$

On the otherhand, Mangontarum *et al* [8] in their attempt to introduce the notion of translated Dowling polynomials and numbers, established further properties for the translated Whitney numbers $\widetilde{W}_{(\alpha)}(n,k)$. The said identities are the *explicit formula*

$$\widetilde{W}_{(\alpha)}(n,k) = \frac{1}{\alpha^k k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}(\alpha i)^n \qquad (0.7)$$

and the *exponential generating function*

$$\left(\frac{e^{\alpha z}-1}{\alpha}\right)^k = k!\sum_{n=k}^{\infty}\widetilde{W}_{(\alpha)}(n,k)\frac{z^n}{n!}. \qquad (0.8)$$

Looking at these properties, we immediately see that

$$\widetilde{w}_{(-1)}(n,k) = s(n,k), \quad \widetilde{W}_{(1)}(n,k) = S(n,k)$$

and

$$\widetilde{W}_{(\alpha)}(n,k) = \alpha^{n-k}S(n,k).$$

These numbers can be shown to be particular cases of the r-*Whitney numbers* $w_{m,r}(n,k)$ *and* $W_{m,r}(n,k)$ *of the first and second kind* defined by Mező [4] via

$$m^n(x)_n = \sum_{k=0}^{n} w_{m,r}(n,k)(mx+r)^k \qquad (0.9)$$

and

$$(mx+r)^n = \sum_{k=0}^{n} m^k W_{m,r}(n,k)(x)_k, \qquad (0.10)$$

respectively. These numbers were further studied by Cheon and Jung [5].

Since fewer attention is given to the translated Whitney numbers of the first kind, it is then among the purpose of these paper to establish some combinatorial properties for the said numbers. To do so, we start by defining

$$w^*_{(\alpha)}(n,k) := (-1)^{k-n}\widetilde{w}_{(\alpha)}(n,k) \tag{0.11}$$

as the *"signed" translated Whitney numbers of the first kind*. Other objectives of this paper are the following:

1. to establish more properties for the numbers $\widetilde{W}_{(\alpha)}(n,k)$, $w^*_{(\alpha)}(n,k)$ and $\widetilde{w}_{(\alpha)}(n,k)$ such as other forms of recurrence relations, some generating functions, orthogonality and inverse relations; and
2. to present identities relating the numbers $\widetilde{W}_{(\alpha)}(n,k)$ and $w^*_{(\alpha)}(n,k)$ with the Bernoulli polynomials which is analogous to the work of Mező in [4].

2. THE SIGNED TRANSLATED WHITNEY NUMBERS OF THE FIRST KIND

Note that the generating function in (0.5) can be expressed as

$$\alpha^n \langle x\rangle_n = \sum_{k=0}^{n}\alpha^k\widetilde{w}_{(\alpha)}(n,k)x^k, \tag{0.12}$$

where $\langle x\rangle_n = x(x+1)(x+2)\cdots(x+n-1)$ is the n-th order rising factorial of x. Replacing x with $-t$ and making use of the fact that

$$\langle -t\rangle_n = (-1)^n(t)_n,$$

gives us

$$\alpha^n(t)_n = \sum_{k=0}^{n}(-1)^{k-n}\widetilde{w}_{(\alpha)}(n,k)(\alpha t)^k. \tag{0.13}$$

Thus, the next theorem is an immediate result.

Theorem 0.1 (horizontal generating function). *The numbers $w^*_{(\alpha)}(n,k)$ satisfy*

$$(t|-\alpha)_n = \sum_{k=0}^{n} w^*_{(\alpha)}(n,k)t^k. \tag{0.14}$$

We can see from (0.14) that when $n = 0$, $\widetilde{w}_{(\alpha)}(0,0) = 1$. By convention, we set $\widetilde{w}_{(\alpha)}(n,k) = 0$ whenever $n < k$ or $n,k < 0$. Also, it will be seen in the next theorem $w^*_{(\alpha)}(n,0) = 0$ for $n > 0$.

Now, since

$$t(t|-\alpha)_n - \alpha n(t|-\alpha)_n = (t|-\alpha)_{n+1},$$

then from (0.14),

$$\begin{aligned}
\sum_{k=0}^{n+1} w^*_{(\alpha)}(n+1,k)t^k &= \sum_{k=0}^{n} w^*_{(\alpha)}(n,k)t^{k+1} - \alpha n\sum_{k=0}^{n} w^*_{(\alpha)}(n,k)t^k \\
&= \sum_{k=0}^{n+1} w^*_{(\alpha)}(n,k-1)t^k - \alpha n\sum_{k=0}^{n+1} w^*_{(\alpha)}(n,k)t^k \\
&= \sum_{k=0}^{n+1}\left\{w^*_{(\alpha)}(n,k-1) - (\alpha n)w^*_{(\alpha)}(n,k)\right\}t^k.
\end{aligned}$$

Comparing the coefficients of t^k yields the triangular recurrence relation in the next theorem.

Theorem 0.2 (recurrence relations). *The numbers $w^*_{(\alpha)}(n,k)$ satisfy the following:*

- *triangular recurrence relation*

$$w^*_{(\alpha)}(n+1,k) = w^*_{(\alpha)}(n,k-1) - (\alpha n)w^*_{(\alpha)}(n,k); \tag{0.15}$$

- *vertical recurrence relation*

$$w^*_{(\alpha)}(n+1,k+1) = \sum_{j=k}^{n}(-\alpha)^{n-k}w^*_{(\alpha)}(j,k)(n)_{n-j}; \tag{0.16}$$

- *horizontal recurrence relation*

$$w^*_{(\alpha)}(n,k) = \sum_{j=0}^{n-k}(\alpha j)^j w^*_{(\alpha)}(n+1,k+j+1). \tag{0.17}$$

Proof. Replacing k by $k+1$ in (0.15) gives

$$w^*_{(\alpha)}(n+1,k+1) = w^*_{(\alpha)}(n,k) - (\alpha n)w^*_{(\alpha)}(n,k+1). \tag{0.18}$$

By applying this repeatedly, we have

$$\begin{aligned} w^*_{(\alpha)}(n+1,k+1) &= w^*_{(\alpha)}(n,k) - \alpha n w^*_{(\alpha)}(n-1,k) + \alpha^2 n(n-1)w^*_{(\alpha)}(n-2,k) - \ldots \\ &\quad +(-\alpha)^{n-k}n(n-1)(n-2)\cdots(n-(n-k)+1)w^*_{(\alpha)}(k+1,k+1). \end{aligned}$$

Since

$$w^*_{(\alpha)}(k+1,k+1) = w^*_{(\alpha)}(k,k),$$

we have

$$\begin{aligned} w^*_{(\alpha)}(n+1,k+1) &= w^*_{(\alpha)}(n,k) - \alpha n w^*_{(\alpha)}(n-1,k) + \alpha^2 n(n-1)w^*_{(\alpha)}(n-2,k) \\ &\quad - \ldots + (-\alpha)^{n-k}n(n-1)(n-2)\cdots(k+1)w^*_{(\alpha)}(k,k) \\ &= \sum_{j=k}^{n}(-\alpha)^{n-j}w^*_{(\alpha)}(j,k)(n)_{n-j}. \end{aligned}$$

For (0.17), we start by solving for $\widetilde{w}_{(\alpha)}(n-1,k-1)$ in (0.18). That is

$$w^*_{(\alpha)}(n,k) = w^*_{(\alpha)}(n+1,k+1) + (\alpha n)w^*_{(\alpha)}(n,k+1). \tag{0.19}$$

Successive application of (0.19) gives

$$\begin{aligned} w^*_{(\alpha)}(n,k) &= w^*_{(\alpha)}(n+1,k+1) + \alpha n\left[w^*_{(\alpha)}(n+1,k+2) + \alpha n w^*_{(\alpha)}(n,k+2)\right] \\ &= w^*_{(\alpha)}(n+1,k+1) + \alpha n w^*_{(\alpha)}(n+1,k+2) + (\alpha n)^2 w^*_{(\alpha)}(n,k+2) \\ &= w^*_{(\alpha)}(n+1,k+1) + \alpha n w^*_{(\alpha)}(n+1,k+2) + (\alpha n)^2 w^*_{(\alpha)}(n,k+3) \\ &\quad + \ldots + (\alpha n)^{n-k}w^*_{(\alpha)}(n+1,n+1) \\ &= \sum_{j=0}^{n-k}(\alpha n)^j w^*_{(\alpha)}(n+1,k+j+1), \end{aligned}$$

which is the desired result.

□

The results in Theorem 0.2 are actually generalizations of the known recurrence relations for the Stirling numbers of the first kind. These relations are very useful in comuting the values for the numbers $w^*_{(\alpha)}(n,k)$. More precisely, using (0.4), the following figures showing the first few values for the numbers $\widetilde{w}_{(\alpha)}(n,k)$ for $\alpha = 2,3$ can be easily obtained.

n/k	0	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0	0	0
2	0	-2	1	0	0	0	0	0	0	0	0
3	0	8	-6	1	0	0	0	0	0	0	0
4	0	-48	44	-12	1	0	0	0	0	0	0
5	0	384	-400	140	-20	1	0	0	0	0	0
6	0	-3840	4384	-1800	340	-30	1	0	0	0	0
7	0	46080	-56448	25984	-5880	700	-42	1	0	0	0
8	0	-645120	836352	-420224	108304	-15680	1288	-56	1	0	0
9	0	9031680	-12354048	6719488	-1936480	327824	-33712	2072	-70	1	0
10	0	-126443520	181988352	-106426880	33830208	-6526016	799792	-62720	3052	-84	1

Figure 1: Some values of $\widetilde{w}_{(2)}(n,k)$

n/k	0	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0	0	0
2	0	-3	1	0	0	0	0	0	0	0	0
3	0	18	-9	1	0	0	0	0	0	0	0
4	0	-162	99	-18	1	0	0	0	0	0	0
5	0	1944	-1350	315	-30	1	0	0	0	0	0
6	0	-29160	22194	-6075	765	-45	1	0	0	0	0
7	0	524880	-428652	131544	-19845	1575	-63	1	0	0	0
8	0	-11022480	9526572	-3191076	548289	-52920	2898	-84	1	0	0
9	0	231472080	-211080492	76539168	-14705145	1659609	-113778	4662	-105	1	0
10	0	-4860913680	4664162412	-1818403020	385347213	-49556934	4048947	-211680	6867	-126	1

Figure 2: Some values of $\widetilde{w}_{(3)}(n,k)$

Identities (0.16) and (0.17) can be best remembered via following figure: As seen in Figure 3,

n/k	$\ldots$	k	$k+1$	$k+2$	$\ldots$	$n+1$
$\vdots$		$\downarrow \quad \downarrow$ $\downarrow \quad \downarrow$				
k		$w^*_{(\alpha)}(k,k)$ $\downarrow \quad \downarrow$ $\downarrow \quad \downarrow$				
$k+1$		$w^*_{(\alpha)}(k+1,k)$ $\downarrow \quad \downarrow$				
$\vdots$		$\downarrow \quad \vdots \quad \downarrow$ $\downarrow \quad \downarrow$				
n		$w^*_{(\alpha)}(n,k)$ $\searrow \quad \nwarrow$ $\searrow \quad \nwarrow$ $\searrow \quad \nwarrow$ $\searrow$				
$n+1$		$\searrow$	$\longleftarrow\longleftarrow\longleftarrow\longleftarrow\longleftarrow$ $w^*_{(\alpha)}(n+1,k+1)$ $\longleftarrow\longleftarrow\longleftarrow\longleftarrow\longleftarrow$	$\longleftarrow\longleftarrow\longleftarrow\longleftarrow\longleftarrow$ $w^*_{(\alpha)}(n+1,k+2)$ $\longleftarrow\longleftarrow\longleftarrow\longleftarrow\longleftarrow$	$\longleftarrow$ $\cdots$ $\longleftarrow$	$\longleftarrow\longleftarrow\longleftarrow\longleftarrow\longleftarrow$ $w^*_{(\alpha)}(n+1,n+1)$ $\longleftarrow\longleftarrow\longleftarrow\longleftarrow\longleftarrow$

Figure 3: Illustration of (0.16) and (0.17)

the values involved in solving for $w^*_{(\alpha)}(n+1,k+1)$ and $w^*_{(\alpha)}(n,k)$ using the vertical and horizontal recurrence relations clearly form a Hockey-stick pattern. Hence, it is safe to say that equations (0.20) and (0.21) are analogous to the *Chu Shih-Chieh's identities for binomial coefficients* (also known as "Hockey-Stick Identities") which can be seen in the book of Chen and Kho [9].

Also, the following corollary is a direct consequence.

Corollary 0.3. *The translated Whitney numbers of the first kind satisfy the following vertical and horizontal recurrence relations:*

$$\widetilde{w}_{(\alpha)}(n+1,k+1) = \sum_{j=k}^{n} \alpha^{n-j} \widetilde{w}_{(\alpha)}(j,k)(n)_{n-j}, \tag{0.20}$$

$$\widetilde{w}_{(\alpha)}(n,k) = \sum_{j=0}^{n-k} (-\alpha n)^j \widetilde{w}_{(\alpha)}(n+1,k+j+1). \tag{0.21}$$

The next theorem presents the exponential generating function for the numbers $w^*_{(\alpha)}(n,k)$. This is important in establishing a relationship between the translated Whitney numbers and the Bernoulli polynomials.

Theorem 0.4 (exponential generating function)**.** *The sequence* $\langle \widetilde{w}^*_{(\alpha)}(n,k) \rangle$ *is generated by*

$$\left(\frac{\log(1+\alpha z)}{\alpha} \right)^k = k! \sum_{n=k}^{\infty} w^*_{(\alpha)}(n,k) \frac{z^n}{n!}. \tag{0.22}$$

Proof. Note that (0.14) can be written as

$$\alpha^n (t)_n = \sum_{k=0}^{n} \alpha^k w^*_{(\alpha)}(n,k) t^k \tag{0.23}$$

Thus,

$$\begin{aligned}
\sum_{k=0}^{\infty} \left\{ \alpha^k \sum_{n=k}^{\infty} w^*_{(\alpha)}(n,k) \frac{z^n}{n!} \right\} t^k &= \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{\infty} \alpha^k w^*_{(\alpha)}(n,k) t^k \right\} \frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty} \alpha^n (t)_n \frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty} \binom{t}{n} (\alpha z)^n \\
&= (1+\alpha z)^t \\
&= \text{exp}\{t \log(1+\alpha z)\}
\end{aligned}$$

Thus,

$$\sum_{k=0}^{\infty} \left\{ \alpha^k \sum_{n=k}^{\infty} w^*_{(\alpha)}(n,k) \frac{z^n}{n!} \right\} t^k = \sum_{k=0}^{\infty} \left\{ \frac{[\log(1+\alpha z)]^k}{k!} \right\} t^k. \tag{0.24}$$

Comparing the coefficient of t^k, gives

$$\alpha^k \sum_{n=k}^{\infty} w^*_{(\alpha)}(n,k) \frac{z^n}{n!} = \frac{[\log(1+\alpha z)]^k}{k!}. \tag{0.25}$$

This is precisely (0.22). □

Note that by definition,

$$\left(\frac{\log(1+\alpha z)}{\alpha} \right)^k = k! \sum_{n=k}^{\infty} (-1)^{k-n} \widetilde{w}_{(\alpha)}(n,k) \frac{z^n}{n!}. \tag{0.26}$$

This can be rewritten as

$$\left(-\frac{\log(1-\alpha(-z))}{\alpha}\right)^k = k!\sum_{n=k}^{\infty}\widetilde{w}_{(\alpha)}(n,k)\frac{(-z)^n}{n!}. \tag{0.27}$$

Replacing $-z$ with u gives us the following corollary:

Corollary 0.5. *The translated Whitney numbers of the first kind satisfy the exponential generating function*

$$\left(-\frac{\log(1-\alpha u)}{\alpha}\right)^k = k!\sum_{n=k}^{\infty}\widetilde{w}_{(\alpha)}(n,k)\frac{u^n}{n!}. \tag{0.28}$$

3. MORE PROPERTIES OF THE TRANSLATED WHITNEY NUMBERS OF THE SECOND KIND

Applying the recurrence relation in (0.4), the following tables are obtained for $\alpha = 2$ and $\alpha = 3$: In this

n/k	0	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0	0	0
2	0	2	1	0	0	0	0	0	0	0	0
3	0	4	6	1	0	0	0	0	0	0	0
4	0	8	28	12	1	0	0	0	0	0	0
5	0	16	120	100	20	1	0	0	0	0	0
6	0	32	496	720	260	30	1	0	0	0	0
7	0	64	2016	4816	2800	560	42	1	0	0	0
8	0	128	8128	30912	27216	8400	1064	56	1	0	0
9	0	256	32640	193600	248640	111216	21168	1848	72	1	0
10	0	512	130816	1194240	2182720	1360800	365232	47040	3000	90	1

Figure 4: Some values of $\widetilde{W}_{(2)}(n,k)$

n/k	0	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0	0	0
2	0	3	1	0	0	0	0	0	0	0	0
3	0	9	9	1	0	0	0	0	0	0	0
4	0	27	63	18	1	0	0	0	0	0	0
5	0	81	405	225	30	1	0	0	0	0	0
6	0	243	2511	2430	585	45	1	0	0	0	0
7	0	729	15309	24381	9450	1260	63	1	0	0	0
8	0	2187	92583	234738	137781	28350	2394	84	1	0	0
9	0	6561	557685	2205225	1888110	563031	71442	4158	108	1	0
10	0	19683	3352671	20404710	24862545	10333575	1848987	158760	6750	135	1

Figure 5: Some values of $\widetilde{W}_{(3)}(n,k)$

section, we establish more combinatorial identities for the translated Whitney numbers of the second kind.

Theorem 0.6 (more recurrence relations)**.** *The translated Whitney numbers of the second kind satisfies the following recurrence relations:*

- *vertical recurrence relation*

$$\widetilde{W}_{(\alpha)}(n+1,k+1) = \sum_{j=k}^{n}[\alpha(k+1)]^{n-j}\widetilde{W}_{(\alpha)}(j,k); \tag{0.29}$$

- *horizontal recurrence relation*

$$\widetilde{W}_{(\alpha)}(n,k)=\sum_{j=0}^{n-k}(-\alpha)^j\prod_{i=1}^{j}(k+i)\widetilde{W}_{(\alpha)}(n+1,k+j+1). \tag{0.30}$$

Proof. Replacing n by $n+1$ and k by $k+1$ in (0.4) gives

$$\widetilde{W}_{(\alpha)}(n+1,k+1)=\widetilde{W}_{(\alpha)}(n,k)+\alpha(k+1)\widetilde{W}_{(\alpha)}(n,k+1). \tag{0.31}$$

Successive application of (0.31) yields

$$\begin{aligned}\widetilde{W}_{(\alpha)}(n+1,k+1) &= \widetilde{W}_{(\alpha)}(n,k)+\alpha(k+1)\widetilde{W}_{(\alpha)}(n-1,k)+[\alpha(k+1)]^2\widetilde{W}_{(\alpha)}(n-2,k)\\ &\quad +\ldots+[\alpha(k+1)]^{n-k}\widetilde{W}_{(\alpha)}(k+1,k+1).\end{aligned}$$

Since

$$\widetilde{w}_{(\alpha)}(k+1,k+1)=\widetilde{w}_{(\alpha)}(k,k),$$

then simplifying the right-hand side gives (0.29). Now, if we express (0.31) as

$$\widetilde{W}_{(\alpha)}(n,k)=\widetilde{W}_{(\alpha)}(n+1,k+1)-\alpha(k+1)\widetilde{W}_{(\alpha)}(n,k+1), \tag{0.32}$$

then successive application of (0.32) yields (0.30). □

Figure 6 illustrates the vertical and horizontal recurrence relations in (0.29) and (0.30).

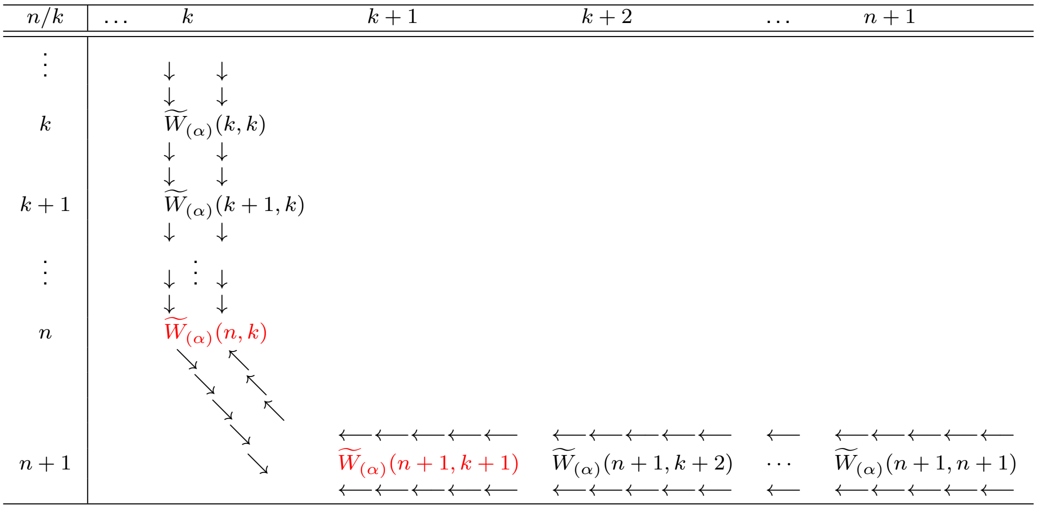

Figure 6: Illustration of (0.29) and (0.30)

Remark 0.1. When $\alpha=1$ in (0.29) and (0.30), the following known identities for the classical Stirling numbers of the second kind are recovered:

$$S(n+1,k+1)=\sum_{j=k}^{n}(k+1)^{n-j}S(j,k); \tag{0.33}$$

$$S(n,k)=\sum_{j=0}^{n-k}(-1)^j\prod_{i=1}^{j}(k+i)S(n+1,k+j+1). \tag{0.34}$$

Theorem 0.7 (rational generating function). *The numbers $\widetilde{W}_{(\alpha)}(n,k)$ satisfy the function*

$$\sum_{n=k}^{\infty} \widetilde{W}_{(\alpha)}(n,k) z^{n-k} = \frac{1}{(1-\alpha z)(1-2\alpha z)(1-3\alpha z)\cdots(1-k\alpha z)}. \tag{0.35}$$

Proof. We will prove this by induction on k. Now, it is easy to verify that (0.35) holds when $k=0$. Hence, we proceed by assuming that (0.35) holds for $k>0$. Applying (0.4),

$$\begin{aligned}
\sum_{n=k+1}^{\infty} \widetilde{W}_{(\alpha)}(n,k+1)z^n &= \sum_{n=k+1}^{\infty} \left\{ \widetilde{W}_{(\alpha)}(n-1,k) + \alpha(k+1)\widetilde{W}_{(\alpha)}(n-1,k+1) \right\} z^n \\
&= z \sum_{n-1=k}^{\infty} \widetilde{W}_{(\alpha)}(n-1,k) z^{n-1} + \alpha z(k+1) \sum_{n-1=k}^{\infty} \widetilde{W}_{(\alpha)}(n-1,k+1) z^{n-1} \\
&= z \left(\frac{z^k}{\prod_{j=0}^{k}(1-\alpha j z)} \right) + \alpha z(k+1) \left(\frac{z^{k+1}}{\prod_{j=0}^{k+1}(1-\alpha j z)} \right) \\
&= \frac{z^{k+1}}{\prod_{j=0}^{k+1}(1-\alpha j z)}.
\end{aligned}$$

This proves the theorem. ☐

Remark 0.2. The well-known classical identity

$$\sum_{n=k}^{\infty} S(n,k) z^{n-k} = \frac{1}{(1-z)(1-2z)(1-3z)\cdots(1-kz)}. \tag{0.36}$$

can be obtained from (0.35) by setting $\alpha = 1$.
An equivalent form of (0.35) is

$$\begin{aligned}
\sum_{n=k}^{\infty} \widetilde{W}_{(\alpha)}(n,k) z^n &= \frac{z^k}{(1-\alpha z)(1-2\alpha z)(1-3\alpha z)\cdots(1-k\alpha z)} \\
&= \frac{1}{\alpha^k} \cdot \frac{(-1)^k}{\left(1-\frac{1}{\alpha z}\right)\left(2-\frac{1}{\alpha z}\right)\left(3-\frac{1}{\alpha z}\right)\cdots\left(k-\frac{1}{\alpha z}\right)} \\
&= \frac{1}{\alpha^k} \cdot \frac{(-1)^k}{\left(\frac{\alpha z-1}{\alpha z}\right)\left(\frac{\alpha z-1}{\alpha z}+1\right)\left(\frac{\alpha z-1}{\alpha z}+2\right)\left(\frac{\alpha z-1}{\alpha z}+3\right)\cdots\left(\frac{\alpha z-1}{\alpha z}+(k+1)\right)} \\
&= \frac{1}{\alpha^k} \cdot \frac{(-1)^k}{\left\langle \frac{\alpha z-1}{\alpha z} \right\rangle_k}.
\end{aligned}$$

Multiplying both sides of this equation by x^k and summing over yields

$$\sum_{k=0}^{\infty} \left\{ \sum_{n=k}^{\infty} \widetilde{W}_{(\alpha)}(n,k) z^n \right\} x^k = \sum_{k=0}^{\infty} \frac{\langle -1 \rangle_k}{\left\langle \frac{\alpha z-1}{\alpha z} \right\rangle_k} \cdot \frac{\left(\frac{-x}{\alpha}\right)^k}{k!}.$$

By the definition of the *translated Dowling polynomials* $\widetilde{D}_{(\alpha)}(n;x)$ [8], viz.

$$\widetilde{D}_{(\alpha)}(n;x) = \sum_{k=0}^{n} \widetilde{W}_{(\alpha)}(n,k) x^k \tag{0.37}$$

and the hypergeometric function (or hypergeometric series) is defined by

$${}_pF_q\left(\begin{matrix} a_1, & a_2, & \ldots, & a_p \\ b_1, & b_2, & \ldots, & b_q \end{matrix} \middle| \, t \right) = \sum_{k=0}^{\infty} \frac{\langle a_1 \rangle_k \langle a_2 \rangle_k \cdots \langle a_p \rangle_k}{\langle b_1 \rangle_k \langle b_2 \rangle_k \cdots \langle b_q \rangle_k} \frac{t^k}{k!}, \tag{0.38}$$

we get

$$\sum_{n=0}^{\infty} \widetilde{D}_{(\alpha)}(n;x)z^n = {}_1F_1\left(\begin{array}{c} 1 \\ \frac{\alpha z-1}{\alpha z} \end{array} \middle| -\frac{x}{\alpha} \right).$$

Finally, the next theorem is obtained by using Kummer's formula seen in M. Abramowitz and I. A. Stegun, eds. [10, p. 505]

$$e^{-x}\, {}_1F_1\left(\begin{array}{c} a \\ b \end{array} \middle| x \right) = {}_1F_1\left(\begin{array}{c} b-a \\ b \end{array} \middle| -x \right).$$

Theorem 0.8. *The translated Dowling polynomials satisfy the generating function*

$$\sum_{n=0}^{\infty} \widetilde{D}_{(\alpha)}(n;x)z^n = \left(\frac{1}{e}\right)^{x/\alpha} {}_1F_1\left(\begin{array}{c} -\frac{1}{\alpha z} \\ \frac{\alpha z-1}{\alpha z} \end{array} \middle| \frac{x}{\alpha} \right). \tag{0.39}$$

Before closing this section, we note that several combinatorial properties of the polynomials $\widetilde{D}_{(\alpha)}(n;x)$ were first investigated by Mangontarum *et al.* [8]. Also, generalizations of (0.39) can be seen in [11] and [12].

4. ORTHOGONALITY AND INVERSE RELATIONS

Theorem 0.9 (orthogonality relations)**.** *The translated Whitney numbers satisfy the following:*

$$\sum_{k=m}^{n} \widetilde{W}_{(\alpha)}(n,k)w^*_{(\alpha)}(k,m) = \sum_{k=m}^{n} w^*_{(\alpha)}(n,k)\widetilde{W}_{(\alpha)}(k,m) = \delta_{mn}, \tag{0.40}$$

where $\delta_{mn} = \begin{cases} 0, & if\ m \neq n \\ 1, & if\ m = n \end{cases}$ *is called "Kronecker's delta".*

Proof. Applying (0.14) to (0.6) gives

$$\begin{aligned} x^n &= \sum_{k=0}^{n} \widetilde{W}_{(\alpha)}(n,k) \sum_{m=0}^{k} w^*_{(\alpha)}(k,m)x^m \\ &= \sum_{m=0}^{n} \left\{ \sum_{k=m}^{n} \widetilde{W}_{(\alpha)}(n,k)w^*_{(\alpha)}(k,m) \right\} x^m. \end{aligned}$$

Comparing the coefficients of x^m and we have

$$\sum_{k=m}^{n} \widetilde{W}_{(\alpha)}(n,k)w^*_{(\alpha)}(k,m) = \begin{cases} 0, & if\ m \neq n \\ 1, & if\ m = n \end{cases}. \tag{0.41}$$

By similar method, we also have

$$\begin{aligned} (x|-\alpha)_n &= \sum_{k=0}^{n} w^*_{(\alpha)}(n,k) \sum_{m=0}^{k} \widetilde{W}_{(\alpha)}(k,m)(x|-\alpha)_m \\ &= \sum_{m=0}^{n} \left\{ \sum_{k=m}^{n} w^*_{(\alpha)}(n,k)\widetilde{W}_{(\alpha)}(k,m) \right\} (x|-\alpha)_m. \end{aligned}$$

Comparing the coefficient of $(x|-\alpha)_m$ completes the proof. □

Since $\widetilde{W}_{(\alpha)}(n,k) = w^*_{(\alpha)}(n,k) = 0$ when $n < k$, then we have

$$\sum_{k=0}^{\infty} \widetilde{W}_{(\alpha)}(n,k) w^*_{(\alpha)}(k,m) = \sum_{k=0}^{\infty} w^*_{(\alpha)}(n,k) \widetilde{W}_{(\alpha)}(k,m) = \delta_{nm}. \tag{0.42}$$

Now, we define $\mathcal{N}_\alpha$ to be an infinite matrix with $\widetilde{W}_{(\alpha)}(i,j)$ as the (i,j)th entries for $i,j = 0,1,2,3,\ldots$ and $\mathcal{M}_\alpha$ as a similar matrix for $w^*_{(\alpha)}(i,j)$. That is

$$\mathcal{M}_\alpha = \begin{pmatrix} w^*_{(\alpha)}(0,0) & 0 & 0 & 0 & \cdots \\ w^*_{(\alpha)}(1,0) & w^*_{(\alpha)}(1,1) & 0 & 0 & \cdots \\ w^*_{(\alpha)}(2,0) & w^*_{(\alpha)}(2,1) & w^*_{(\alpha)}(2,2) & 0 & \cdots \\ w^*_{(\alpha)}(3,0) & w^*_{(\alpha)}(3,1) & w^*_{(\alpha)}(3,2) & w^*_{(\alpha)}(3,3) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$\mathcal{N}_\alpha = \begin{pmatrix} \widetilde{W}_{(\alpha)}(0,0) & 0 & 0 & 0 & 0 & \cdots \\ \widetilde{W}_{(\alpha)}(1,0) & \widetilde{W}_{(\alpha)}(1,1) & 0 & 0 & 0 & \cdots \\ \widetilde{W}_{(\alpha)}(2,0) & \widetilde{W}_{(\alpha)}(2,1) & \widetilde{W}_{(\alpha)}(2,2) & 0 & 0 & \cdots \\ \widetilde{W}_{(\alpha)}(3,0) & \widetilde{W}_{(\alpha)}(3,1) & \widetilde{W}_{(\alpha)}(3,2) & \widetilde{W}_{(\alpha)}(3,3) & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Then, by (0.42), we have

$$\mathcal{N}_\alpha \cdot \mathcal{M}_\alpha = \mathcal{M}_\alpha \cdot \mathcal{N}_\alpha = \mathcal{I},$$

where $\mathcal{I}$ is the infinite-dimensional identity matrix. Thus, we may conclude that

$$\mathcal{M}_\alpha = \mathcal{N}_\alpha^{-1},$$

where $\mathcal{N}_\alpha^{-1}$ is the inverse of $\mathcal{N}_\alpha$ and vice versa. In the following corollary, we present the orthogonality relations for the translated Whitney numbers of the first kind.

Corollary 0.10. *The following relations hold:*

$$\sum_{k=m}^{n} (-1)^{m-k} \widetilde{W}_{(\alpha)}(n,k) \widetilde{w}_{(\alpha)}(k,m) = \sum_{k=m}^{n} (-1)^{k-n} \widetilde{w}_{(\alpha)}(n,k) \widetilde{W}_{(\alpha)}(k,m) = \delta_{mn}. \tag{0.43}$$

In the next theorem, we present other inverse relations for the numbers $\widetilde{w}^*_{(\alpha)}(n,k)$ and $\widetilde{W}_{(\alpha)}(n,k)$.

Theorem 0.11 (inverse relations)**.** *The translated Whitney numbers satisfy the following:*

$$f_n = \sum_{k=0}^{n} w^*_{(\alpha)}(n,k) g_k \iff g_n = \sum_{k=0}^{n} \widetilde{W}_{(\alpha)}(n,k) f_k, \tag{0.44}$$

and

$$f_k = \sum_{n=k}^{\infty} w^*_{(\alpha)}(n,k) g_n \iff g_k = \sum_{n=k}^{\infty} \widetilde{W}_{(\alpha)}(n,k) f_n. \tag{0.45}$$

Proof. Using the hypothesis,

$$\begin{aligned} \sum_{k=0}^{n} \widetilde{W}_{(\alpha)}(n,k) f_k &= \sum_{k=0}^{n} \widetilde{W}_{(\alpha)}(n,k) \sum_{m=0}^{k} w^*_{(\alpha)}(k,m) g_m \\ &= \sum_{m=0}^{n} \left\{ \sum_{k=m}^{n} \widetilde{W}_{(\alpha)}(n,k) w^*_{(\alpha)}(k,m) \right\} g_m. \end{aligned}$$

Applying (0.40),

$$\sum_{k=0}^{n}\widetilde{W}_{(\alpha)}(n,k)f_k=\sum_{m=0}^{n}\delta_{mn}g_m=g_n. \tag{0.46}$$

The proof of the converse of (0.44) is similar. For (0.45), we have

$$\begin{aligned}\sum_{k=0}^{\infty}\left\{\sum_{n=k}^{\infty}\widetilde{W}_{(\alpha)}(n,k)f_n\right\}(x|-\alpha)_k &= \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{\infty}\widetilde{W}_{(\alpha)}(n,k)(x|-\alpha)_k\right\}f_n\\ &= \sum_{m=0}^{\infty}\left\{\sum_{n=0}^{\infty}w^*_{(\alpha)}(m,n)\sum_{k=0}^{\infty}\widetilde{W}_{(\alpha)}(n,k)(x|-\alpha)_k\right\}g_m\\ &= \sum_{m=0}^{\infty}\left\{\sum_{k=0}^{\infty}\left\{\sum_{n=0}^{\infty}w^*_{(\alpha)}(m,n)\widetilde{W}_{(\alpha)}(n,k)\right\}(x|-\alpha)_k\right\}g_m.\end{aligned}$$

(0.40) gives us

$$\begin{aligned}\sum_{k=0}^{\infty}\left\{\sum_{n=k}^{\infty}\widetilde{W}_{(\alpha)}(n,k)f_n\right\}(x|-\alpha)_k &= \sum_{m=0}^{\infty}\left\{\sum_{k=0}^{\infty}\delta_{km}(x|-\alpha)_k\right\}g_m\\ &= \sum_{k=0}^{\infty}\left\{\sum_{m=k}^{\infty}\delta_{km}g_m\right\}(x|-\alpha)_k\\ &= \sum_{k=0}^{\infty}\left\{\delta_{kk}g_k+\delta_{kk+1}g_{k+1}+\cdots\right\}(x|-\alpha)_k\\ &= \sum_{k=0}^{\infty}\left\{g_k\right\}(x|-\alpha)_k.\end{aligned}$$

By comparing the coefficients of $(x|-\alpha)_k$ and we have

$$\sum_{n=k}^{\infty}\widetilde{W}_{(\alpha)}(n,k)f_n=g_k.$$

From here, the converse can be deduced. □

5. AN APPLICATION TO THE BERNOULLI POLYNOMIALS

The *Bernoulli polynomials* $B_n(x)$ is known to be defined by the exponential generating function [4]

$$\sum_{n=0}^{\infty}B_n(x)\frac{z^n}{n!}=\frac{ze^{zx}}{e^z-1}, \tag{0.47}$$

where $B_n(0)=B_n$ are the *Bernoulli numbers*. A pair of interesting identities relating these polynomials with the r-Whitney numbers were obtained by Mező [4]. To be precise, we have the following:

$$\binom{n+1}{l}B_{n-l+1}=\frac{n+1}{m^{n-l+1}}\sum_{k=0}^{n}W_{m,r}(n,k)\frac{w_{m,r}(k+1,l)}{k+1}; \tag{0.48}$$

$$\binom{n+1}{l}B_{n-l+1}(r/m)=\frac{n+1}{m^n}\sum_{k=0}^{n}\frac{m^k}{k+1}W_{m,r}(n,k)s(k+1,l). \tag{0.49}$$

To obtain an identity that is analogous to (0.48), we first rewrite the exponential generating function in (0.22) as

$$\sum_{k=0}^{\infty} \frac{w^*_{(\alpha)}(k+1,l)}{k+1} \cdot \frac{z^k}{k!} = \frac{1}{z} \cdot \frac{[\log(1+\alpha z)]^l}{\alpha^l l!}. \tag{0.50}$$

Next, we combine this with the exponential generating function in (0.8). That is,

$$\begin{aligned}
\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \widetilde{W}_{(\alpha)}(n,k) \frac{w^*_{(\alpha)}(k+1,l)}{k+1} \right) \frac{z^n}{n!} &= \sum_{k=0}^{\infty} \left(\sum_{n=k}^{\infty} \widetilde{W}_{(\alpha)}(n,k) \frac{z^n}{n!} \right) \frac{w^*_{(\alpha)}(k+1,l)}{k+1} \\
&= \sum_{k=0}^{\infty} \frac{w^*_{(\alpha)}(k+1,l)}{k+1} \cdot \frac{\left(\frac{e^{\alpha z}-1}{\alpha}\right)^k}{k!} \\
&= \frac{z^{l-1}}{l!} \cdot \frac{\alpha z}{e^{\alpha z}-1}.
\end{aligned}$$

From (0.47), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \widetilde{W}_{(\alpha)}(n,k) \frac{w^*_{(\alpha)}(k+1,l)}{k+1} \right) \frac{z^n}{n!} &= \frac{z^{l-1}}{l!} \left(\sum_{n=0}^{\infty} B_n \frac{(\alpha z)^n}{n!} \right) \\
&= \sum_{n=l-1}^{\infty} \left\{ B_{n-l+1} \binom{n+1}{l} \frac{\alpha^{n-l+1}}{n+1} \right\} \frac{z^n}{n!}.
\end{aligned}$$

The result obtained when we compare the coefficients of z^n is stated in the next theorem.

Theorem 0.12. *The numbers $\widetilde{W}_{(\alpha)}(n,k)$ and $w^*_{(\alpha)}(n,k)$ satisfy*

$$\binom{n+1}{l} B_{n-l+1} = \frac{n+1}{\alpha^{n-l+1}} \sum_{k=0}^{n} \widetilde{W}_{(\alpha)}(n,k) \frac{w^*_{(\alpha)}(k+1,l)}{k+1} \tag{0.51}$$

and

$$\binom{n+1}{l} B_{n-l+1} = \frac{n+1}{\alpha^n} \sum_{k=0}^{n} \frac{\alpha^k}{k+1} \widetilde{W}_{(\alpha)}(n,k) s(k+1,l). \tag{0.52}$$

Proof. The proof of (0.52) is similar to (0.51) except that we use the classical known generating function

$$\sum_{k=0}^{\infty} \frac{s(k+1,l)}{k+1} \cdot \frac{z^k}{k!} = \frac{1}{z} \cdot \frac{[\log(1+z)]^l}{l!} \tag{0.53}$$

in place of (0.50). □

Note that when $\alpha = 1$ in equations (0.51) and (0.52), the classical identity in [13] given by

$$\binom{n+1}{l} B_{n-l+1} = (n+1) \sum_{k=0}^{n} S(n,k) s(k+1,l) \frac{1}{k+1}$$

is obtained.

As closing, we recall that the translated Whitney-Lah numbers, denoted by $\widehat{w}_{(\alpha)}(n,k)$, were defined in [6] as the number of ways to distribute the set $\{1,2,3,\ldots,n\}$ into k ordered lists such that the elements in each list can mutate with α ways except the dominant one. These numbers are known to satisfy the following identities (see [6]):

- triangular recurrence relation

$$\widehat{w}_{(\alpha)}(n,k) = \widehat{w}_{(\alpha)}(n-1,k-1) + \alpha(n+k-1)\widehat{w}_{(\alpha)}(n-1,k); \tag{0.54}$$

- horizontal generating function

$$(x|\alpha)_n = \sum_{k=0}^{n} \widehat{w}_{(\alpha)}(n,k)(x| - \alpha)_k; \quad (0.55)$$

- relation

$$\widehat{w}_{(\alpha)}(n,k) = \sum_{j=k}^{n} \widetilde{w}_{(\alpha)}(n,j)\widetilde{W}_{(\alpha)}(j,k). \quad (0.56)$$

The authors would like to direct the attention of the readers to these numbers. Is it possible to establish several combinatorial properties for the Whitney-Lah numbers parallel to the results in this paper? Perhaps answers can be found by examining the properties of the classical Lah numbers given by

$$\frac{n!}{k!}\binom{n-1}{k-1} \quad (0.57)$$

and

$$(-1)^{n-k}\frac{n!}{k!}\binom{n-1}{k-1}. \quad (0.58)$$

6. CONCLUSION

In this paper we have defined the "signed" translated Whitney numbers $w^*_{(\alpha)}(n,k)$ which opened a path to establishing some properties for the translated Whitney numbers of the first. The combinatorial properties obtained in this paper such as the vertical and horizontal recurrence relations, the exponential and rational generating functions and the orthogonality and inverse relations further develops the study of the translated Whitney numbers and more future applications. Moreover, we were able to derive interesting identities relating the translated Whitney numbers (the signed and the second kind) with the well-celebrated Bernoulli polynomials. Though the said identities are particular cases of the results of Mező in [4], they appear to be as compelling as the first work.

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COMPETING INTERESTS

The authors declare that no competing interests exist.

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