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# SUPER $(a, d)$-$C_4$-ANTIMAGICNESS OF BOOK GRAPHS

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ABSTRACT. Let $G = (V, E)$ be a finite simple graph with $|V(G)|$ vertices and $|E(G)|$ edges. An *edge-covering* of $G$ is a family of subgraphs $H_1, H_2, \dots, H_t$ such that each edge of $E(G)$ belongs to at least one of the subgraphs $H_i$, $i = 1, 2, \dots, t$. If every subgraph $H_i$ is isomorphic to a given graph $H$, then the graph $G$ admits an *$H$-covering*. A graph $G$ admitting $H$ covering is called an $(a, d)$-$H$-antimagic if there is a bijection $f : V \cup E \to \{1, 2, \dots, |V(G)| + |E(G)|\}$ such that for each subgraph $H'$ of $G$ isomorphic to $H$, the sum of labels of all the edges and vertices belonged to $H'$ constitutes an arithmetic progression with the initial term $a$ and the common difference $d$. For $f(V) = \{1, 2, 3, \dots, |V(G)|\}$, the graph $G$ is said to be *super $(a, d)$-$H$-antimagic* and for $d = 0$ it is called *$H$-supermagic*. In this paper, we investigate the existence of super $(a, d)$-$C_4$-antimagic labeling of book graphs, for difference $d = 0, 1$ and $n \geq 2$.

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## 1. Introduction

An *edge-covering* of finite and simple graph $G$ is a family of subgraphs $H_1, H_2, \dots,$ $H_t$ such that each edge of $E(G)$ belongs to at least one of the subgraphs $H_i$, $i = 1, 2, \dots, t$. In this case we say that $G$ admits an $(H_1, H_2, \dots, H_t)$-*(edge) covering*. If every subgraph $H_i$ is isomorphic to a given graph $H$, then the graph $G$ admits an *$H$-covering*. A graph $G$ admitting an $H$-covering is called $(a, d)$-*$H$-antimagic*

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if there exists a total labeling $f : V(G) \cup E(G) \rightarrow \{1, 2, \dots, |V(G)| + |E(G)|\}$ such that for each subgraph $H'$ of $G$ isomorphic to $H$, the $H'$-weights,

$$wt_f(H') = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e),$$

constitute an arithmetic progression $a, a+d, a+2d, \dots, a+(t-1)d$, where $a > 0$ and $d \geq 0$ are two integers and $t$ is the number of all subgraphs of $G$ isomorphic to $H$. Moreover, $G$ is said to be *super* $(a, d)$-$H$*-antimagic*, if the smallest possible labels appear on the vertices. If $G$ is a (super) $(a, d)$-$H$-antimagic graph then the corresponding total labeling $f$ is called the *(super)* $(a, d)$-$H$*-antimagic labeling*. For $d = 0$, the (super) $(a, d)$-$H$-antimagic graph is called *(super)* $H$*-magic*.

The (super) $H$-magic graph was first introduced by Gutiérrez and Lladó in [1]. They proved that the star $K_{1,n}$ and the complete bipartite graphs $K_{n,m}$ are $K_{1,h}$-supermagic for some $h$. They also proved that the path $P_n$ and the cycle $C_n$ are $P_h$-supermagic for some $h$. Lladó and Moragas [2] investigated $C_n$-(super)magic graphs and proved that wheels, windmills, books and prisms are $C_h$-magic for some $h$. Some results on $C_n$-supermagic labelings of several classes of graphs can be found in [3]. Maryati *et al.* [4] gave $P_h$-(super)magic labelings of shrubs, subdivision of shrubs and banana tree graphs. Other examples of $H$-supermagic graphs with different choices of $H$ have been given by Jeyanthi and Selvagopal in [5]. Maryati *et al.* [6] investigated the $G$-supermagicness of a disjoint union of $c$ copies of a graph $G$ and showed that disjoint union of any paths is $cP_h$-supermagic for some $c$ and $h$.

The $(a, d)$-$H$-antimagic labeling was introduced by Inayah *et al.* [7]. In [8] Inayah *et al.* investigated the super $(a, d)$-$H$-antimagic labelings for some shackles of a connected graph $H$.

For $H \cong K_2$, (super) $(a, d)$-$H$-antimagic labelings are also called (super) $(a, d)$-edge-antimagic total labelings. For further information on (super) edge-magic labelings, one can see [9, 10, 11, 12].

The (super) $(a, d)$-$H$-antimagic labeling is related to a (super) $d$-antimagic labeling of type $(1, 1, 0)$ of a plane graph which is the generalization of a face-magic labeling introduced by Lih [13]. Further information on super $d$-antimagic labelings can be found in [14, 15, 16].

In this paper, we study the existence of super $(a, d)$-$C_4$-antimagic labeling of book graphs.

## 2. Super $C_4$-antimagic labeling of book graphs and disjoint union of book graphs

In this section, we discuss super $(a, 1)$-$C_4$-antimagicness of *book graphs* for difference $d = 1$ and super $(b, 0)$-$C_4$-antimagicness of disjoint union of book graphs $mB_n$ .

Let $K_{1,n}$, $n \geq 2$ be a complete bipartite graph on $n+1$ vertices. The *book graph* $B_n$ is a cartesian product of $K_{1,n}$ with $K_2$. i.e., $B_n \cong K_{1,n} \square K_2$. Clearly book

graph $B_n$ admits covering by cycle $C_4$.
The book graph $B_n$ has the vertex and edge set

$$V(B_n) = \{y_1, y_2\} \cup \cup_{i=1}^{n} \{x_{(1,i)}, x_{(2,i)}\}$$
$$E(B_n) = \cup_{i=1}^{n} \{y_1 x_{(1,i)}, y_2 x_{(2,i)}, x_{(1,i)} x_{(2,i)}\} \cup \{y_1 y_2\}$$

respectively.
It can be noted that $|V(B_n)| = 2(n+1)$ and $|E(B_n)| = 3n+1$.
Every $C_4^{(j)}, 1 \le j \le n$ in $B_n$ has the vertex set:

$$V(C_4^{(j)}) = \{y_1, y_2, x_{(1,j)}, x_{(2,j)}\}$$

and the edge set is:

$$E(C_4^{(j)}) = \{y_1 y_2, y_1 x_{(1,j)}, y_2 x_{(2,j)}, x_{(1,j)} x_{(2,j)}\}.$$

Under a total labeling $\alpha$, the $C_4^{(j)}$ weights, $j = 1, \dots, n$, would be:

$$\begin{aligned} wt_\alpha(C_4^{(j)}) &= \sum_{v \in V(C_4^{(j)})} \alpha(v) + \sum_{e \in E(C_4^{(j)})} \alpha(e). \\ &= \sum_{k=1}^{2} \alpha(y_k) + \alpha(y_1 y_2) + \sum_{k=1}^{2} \alpha(x_{(k,j)}) + \sum_{k=1}^{2} \alpha(y_k x_{(k,j)}) + \alpha(x_{(1,j)} x_{(2,j)}) \end{aligned} \quad (1)$$

**Theorem 2.1.** *For any integer $n \ge 2$, the book graph $B_n$ is super $(a,1)$-$C_4$-antimagic.*

*Proof.* Under a labeling $\alpha$, the set $\{y_1, y_2, y_1 y_2\}$, would be labeled as as:

$$\alpha(y_1) = 1, \qquad \alpha(y_2) = 2$$
$$\alpha(y_1 y_2) = 2(n+1) + 1,$$

and therefore the partial sum of $wt_\alpha(C_4^{(j)})$ would be

$$\alpha(y_1) + \alpha(y_2) + \alpha(y_1 y_2) = 2(n+3). \quad (2)$$

For remaining set of vertices and edges, the labeling $\alpha$ is defined as:

$$\begin{aligned} \alpha(x_{(t,i)}) &= 2 + j + (k-1)n, && \text{if } k = 1, 2 \\ & && \quad j = 1, 2, \dots, n \\ \alpha(x_{(1,i)} x_{(2,i)}) &= 2(n+1) + 1 + j, && \text{if } j = 1, 2, \dots, n \\ \alpha(y_k x_{(k,i)}) &= (6-k)n + 4 - j, && \text{if } k = 1, 2 \\ & && \quad j = 1, 2, \dots, n \end{aligned}$$

Clearly

$$\alpha(V(B_n)) = \{1, 2, \dots, 2(n+1)\}.$$

Therefore $\alpha$ is a super labeling and together with

$$\alpha(E(B_n)) = \{2(n+1)+1, 2(n+1)+2, \ldots, 5n+3\}$$

it shows $\alpha$ is a total labeling.
Using (1) and (2), $wt_\alpha(C_4^{(j)})$ are:

$$\begin{aligned} wt_\alpha(C_4^{(j)}) &= 2(n+3) + (4+2i+n) + (11n+11-j) \\ &= 14n+21+j. \end{aligned}$$

Thus $wt_\alpha(C_4^{(j)})$ constitute an arithmetic progression with $a = 14n+22$ and $d = 1$. Hence book graphs are super $(a,1)$-$C_4$-antimagic.
This completes the proof. $\square$

**Theorem 2.2.** *For any integer $n \geq 2$ and $n \equiv 1$ (mod 2), the book graph $B_n$ is $C_4$-supermagic.*

*Proof.* Under a labeling $\psi$, the set $\{y_1, y_2, y_1y_2\}$, would be labeled as as:

$$\psi(y_1) = 1, \qquad \psi(y_2) = 2$$
$$\psi(y_1y_2) = 2(n+1)+1,$$

and therefore the partial sum of $wt_\psi(C_4^{(j)})$ would be

$$\psi(y_1) + \psi(y_2) + \psi(y_1y_2) = 2(n+3). \tag{3}$$

For remaining set of vertices and edges, the labeling $\psi$ is defined as:

$$\psi(x_{(k,i)}) = \begin{cases} 2+i, & k=1, i=1,2,\ldots,n \\ 2n+3-i, & k=2, i=1,2,\ldots,n \end{cases}$$

$$\psi(x_{(1,i)}x_{(2,i)}) = 5n+4-i$$

$$\psi(y_kx_{(k,i)}) = \begin{cases} 2n+3+\frac{i+1}{2} & k=1, i=1,3,\ldots,n \\ \frac{5n+7}{2}+\frac{i}{2} & k=1, i=2,4,\ldots,n-1, \\ \frac{7n+6+i}{2} & k=2, i=1,3,\ldots,n \\ 3(n+1)+\frac{i}{2} & k=1, i=2,4,\ldots,n-1 \end{cases}$$

Clearly

$$\psi(V(B_n)) = \{1, 2, \ldots, 2(n+1)\}.$$

Therefore $\psi$ is a super labeling and together with

$$\psi(E(B_n)) = \{2(n+1)+1, 2(n+1)+2, \ldots, 5n+3\}$$

it shows $\psi$ is a total labeling.
Using (1) and (3), the $wt_\psi C_4^{(j)}$ are:

$$\begin{aligned} wt_\psi(C_4^{(j)}) &= 2(n+3) + \left(\frac{11n+13}{2} + i\right) + (5n+4-i) + (2n+5) \\ &= \frac{29n+43}{2} \end{aligned}$$

Thus $wt_{\psi}(C_4^{(j)})$ are independent of $i$. Hence book graphs are $C_4$-supermagic for $n \equiv 1 \pmod 2$. This completes the proof. $\square$

**Theorem 2.3.** *Let $m \geq 1, n \geq 2$ be positive integers and book graph $B_n$ admits a $C_4$-supermagic labeling. Then the disjoint union of arbitrary number of copies of $B_n$, i.e. $mB_n$, also admits a $C_4$-supermagic labeling.*

*Proof.* Let $m$ be a positive integer. By the symbol $x_i, i = 1, 2, \ldots, m$, we denote an element (a vertex or an edge) in the $i^{th}$ copy of the book graph $B_n$, denoted by $B_n(i)$, corresponding to the element $x$ in $B_n$, i.e., $x \in V(B_n) \cup E(B_n)$. Analogously, let $C_4^j(i), i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n$, be the subgraph in the $i^{th}$ copy of $B_n$ corresponding to the subgraph $C_4^j$ in $B_n$.
Let us define the total labeling $\phi$ of $mB_n$ in the following way:

$$\phi(x_i) = \begin{cases} m(\psi(x) - 1) + i & \text{if } x \in V(B_n), \\ m\psi(x) + 1 - i & \text{if } x \in E(B_n). \end{cases}$$

First we shall show that the vertices of $\bigcup_{i=1}^{m} B_n(i)$ under the labeling $\phi$ use integers from 1 up to $pm$, i.e. if $i = 1$ then the vertices of $B_n(1)$ successively attain values $[1, m+1, 2m+1, \ldots, (p-1)m+1]$, if $i = 2$ then the vertices of $B_n(2)$ successively assume values $[2, m+2, 2m+2, \ldots, (p-1)m+2], \ldots,$ the values of vertices of $B_n(i)$ are equal successively to $[i, m+i, 2m+i, \ldots, (p-1)m+i], \ldots,$ if $i = m$ then the vertices of $B_n(m)$ successively assume values $[m, 2m, 3m, \ldots, pm]$.
Second we can see that the edges of $\bigcup_{i=1}^{m} B_n(i)$ under the labeling $\phi$ use integers from $pm+1$ up to $(p+q)m$. It means, if $i = 1$ then the edges of $B_n(1)$ successively assume values $[(p+1)m, (p+2)m, (p+3)m, \ldots, (p+q)m]$, if $i = 2$ then the edges of $B_n(2)$ successively assume values $[(p+1)m-1, (p+2)m-1, (p+3)m-1, \ldots, (p+q)m-1], \ldots,$ the values of edges of $B_n(i)$ are equal successively to $[(p+1)m+1-i, (p+2)m+1-i, (p+3)m+1-i, \ldots, (p+q)m+1-i], \ldots,$ if $i = m$ then the edges of $B_n(m)$ successively assume values $[pm+1, (p+1)m+1, (p+2)m+1, \ldots, (p+q-1)m+1]$.
It is not difficult to see that the labeling $\phi$ is a bijection between the integers $\{1, 2, \ldots, (p+q)m\}$ and the vertices and edges of $\bigcup_{i=1}^{m} B_n(i)$, therefore $\phi$ is a total labeling.
Under the labeling $\phi$, the weights of every subgraph $C_4^{(j)}(i), 1 \leq i \leq m, 1 \leq j \leq k$, where $k$ is the number of $C_4$'s in $B_n(i)$, would be:

$$\begin{aligned} wt_{\phi}(C_{(4,i)}^{(j)}) &= \sum_{v \in V(C_4^{(j)}(i))} \phi(v) + \sum_{e \in E(C_4^{(j)}(i))} \phi(e) \\ &= \sum_{v \in V(C_4^{(j)}(i))} (m(\psi(v) - 1) + i) + \sum_{e \in E(C_4^{(j)}(i))} (m\psi(e) + 1 - i) \\ &= m \sum_{v \in V(C_4^{(j)}(i))} \psi(v) - m|V(C_4^{(j)}(i))| + i|V(C_4^{(j)}(i))| \end{aligned}$$

$$+m\sum_{e\in E(C_4^{(j)}(i))}\psi(e)+|E(C_4^{(j)}(i))|-i|E(C_4^{(j)}(i))|$$

$$=m\left(\sum_{v\in V(C_4^{(j)}(i))}\psi(v)+\sum_{e\in E(C_4^{(j)}(i))}\psi(e)\right)-m|V(C_4^{(j)}(i))|+|E(C_4^{(j)}(i))|$$

$$+i|V(C_4^{(j)}(i))|-i|E(C_4^{(j)}(i))|$$

$$=mwt_{\psi}(C_4^{(j)}(i))-m|V(C_4^{(j)}(i))|+|E(C_4^{(j)}(i))|+i|V(C_4^{(j)}(i))|-i|E(C_4^{(j)}(i))|.$$

As every $C_4^{(j)}(i)$, $i=1,2,\ldots,m, j=1,2,\ldots,k$, is isomorphic to the cycle $C_4$ it holds

$$|V(C_4^{(j)}(i))|=|V(C_4)|=4,$$
$$|E(C_4^{(j)}(i))|=|E(C_4)|=4.$$

Thus for the $C_4$-weights we get

$$\begin{aligned}wt_{\phi}(C_4^{(j)}(i))&=mwt_{\psi}(C_4^{(j)})+4(1-m)\\&=\frac{m}{2}(29n+43)+4(1-m)\\&=\frac{m}{2}(29n+35)+4.\end{aligned}$$

It is easy to see that the set of all $C_4^{(j)}(i)$-weights in $\bigcup_{i=1}^{m}B_n(i)$ consists of same integers. Thus the graph $\bigcup_{i=1}^{m}B_n$ is a $C_4$-supermagic.
This completes the proof. □

## Competing Interests

The authors declare that they have no competing interests.

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