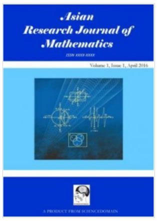

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On Chebyshev and Riemann-Liouville Fractional Inequalities in q-Calculus

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

This paper presents some new inequalities on Fractional calculus in the context of q-calculus. Fractional calculus generalizes the integer order differentiation and integration to derivatives and integrals of arbitrary order. In other words, Fractional calculus explores integrals and derivatives of functions that involve non-integer order(s). Quantum calculus (q-Calculus) on the other hand focuses on investigations related to calculus without limits and in recent times, it has attracted the interest of many researchers due to its high demand of mathematics to model complex systems in nature with anomalous dynamics. This paper thus establishes some new extensions of Chebyshev and Riemann-Liouville fractional integral inequalities for positive and increasing functions via q-Calculus.

Keywords: Chebyshev inequality; riemann-liouville; fractional calculus; q-Calculus.

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1 Introduction

The popular Chebyshev inequality reads that [1]

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \geq \left[\frac{1}{b-a}\int_a^b f(x)dx\right]\left[\frac{1}{b-a}\int_a^b g(x)dx\right], \tag{1.1}$$

where f and g are two integrable and synchronous functions on $[a, b]$.

Two functions f and g are said to be synchronous on $[a, b]$ if

$$[f(x) - f(y)]\,[g(x) - g(y)] \geq 0,$$

for all $x, y \in [a, b]$.

The inequality (1.1) is very important in literature and has many applications in diverse research areas such as numerical quadrature, transform theory, probability, existence of solutions of differential equations, and statistical problems. Many authors have investigated, generalized, and applied this Chebyshev inequality. See ([2, 3, 4, 5] and the references cited therein) for more details.

2 Preliminaries

This section presents some definitions and theorems that are essential to the understanding and establishment of our main results.

Definition 2.1. [6], the q- derivative of f is defined as

$$D_q f(x := \frac{d_q f(x)}{d_q x} = \frac{f(qx) - f(x)}{(q-1)x}, \tag{2.1}$$

where D_q is a linear operator and the q-diferential of f is given by

$$d_q f(x) = f(qx) - f(x). \tag{2.2}$$

Thus

$$D_q(af(x) + bg(x)) = \frac{af(qx) + bg(qx) - af(x) - bg(x)}{(q-1)x}. \tag{2.3}$$

$$D_q\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)D_q f(x) - f(x)D_q g(x)}{g(x)g(qx)}. \tag{2.4}$$

$$D_q\left(f(x)g(x)\right) = g(x)D_q f(x) + f(qx)D_q g(x). \tag{2.5}$$

(See also [7, 8, 9]).

The q-definite integral is defined as

$$\int_a^b f(x)d_q x = \int_0^b f(x)d_q x - \int_0^a f(x)d_q x. \tag{2.6}$$

and

$$\int_0^x f(t)d_q t = (1-q)x\sum_{j=0}^{\infty} q^j f(q^j x), \qquad x \in (0, \infty). \tag{2.7}$$

The q-integration by parts is defined as

$$\int_a^b f(x)(D_q g)(x)d_q x = f(b)g(b) - f(a)g(a) - \int_a^b g(qx)(D_q f)(x)d_q x. \tag{2.8}$$

Here are some further Definitons also found in ([10, 11, 12, 13]).

Definition 2.2. Let $[a, b]$ be a finite interval on the real axis $\mathbb{R}$ for $(-\infty < a < b < \infty)$. Then, the Riemann-Liouville fractional integrals (left-sided) of a function f of order $\alpha \in \mathbb{C}$ with $\mathbb{R}(\alpha) > 0$ is defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad x > a, \tag{2.9}$$

where the Euler Gamma function is given by

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt, \tag{2.10}$$

for $\alpha \in \mathbb{C}$.

The q-Euler Gamma function is defined as

$$\Gamma_q(\alpha) = \int_0^{\frac{1}{q-1}} e^{-qt} t^{\alpha-1} d_q t. \tag{2.11}$$

Details can be found in the following works ([14] and [15] and the references cited therein).

[16] also studied the Riemann-Liouville inequalities and established the following theorems.

Theorem 2.1. *Let f and g be two synchronous functions on $[0, \infty)$, then*

$$J^{\alpha} fg(t) = \frac{\Gamma(a+1)}{t^{\alpha}} J^{\alpha} f(t) J^{\alpha} g(t), \tag{2.12}$$

for $t, \alpha > 0$.

Theorem 2.2. *Let f and g be two synchronous functions on $[0, \infty)$, then*

$$\frac{t^{\alpha}}{\Gamma(\alpha+1)} J^{\beta}(fg)(t) + \frac{t^{\beta}}{\Gamma(\beta+1)} J^{\alpha}(fg)(t) \geq J^{\alpha} f(t) J^{\beta} g(t) + J^{\beta} f(t) J^{\alpha} g(t), \tag{2.13}$$

for $t, \alpha > 0$.

Theorem 2.3. *Let f_i for $1 \leq i \leq n$ be positive and increasing functions on $[0, \infty)$, then*

$$J^{\alpha}(\prod_{i=1}^{n} f_i)(t) \geq [J^{\alpha}(1)]^{1-n} \prod_{i=1}^{n} I^{\alpha} f_i(t), \tag{2.14}$$

for $t, \alpha > 0$.

Theorem 2.4. *Let f and g be two functions defined on $[0, \infty)$ such that f is increasing and g is differentiable and there exist a real number $m = inf_{t \geq 0}\ g'(t)$, then the inequality*

$$J^{\alpha}(fg)(t) \geq \frac{1}{J^{\alpha}(1)} J^{\alpha} f(t) J^{\alpha} g(t) - \frac{mt}{\alpha+1} J^{\alpha} f(t) + m J^{\alpha}(t f(t)), \tag{2.15}$$

is valid for $t, \alpha > 0$.

For the proofs of the theorems, see [16].

The following are some properties of the Riemann-Liouville fractional integrals ([17, 18, 19]).

Let $\alpha, \beta \in \mathbb{R}$ and $f_1 \in C[a,b]$, then

$$I_{a+}^{\beta} I_{a+}^{\alpha} f(x) = I_{a+}^{\alpha} I_{a+}^{\beta} f(x) = I_{a+}^{\alpha+\beta} f(x). \tag{2.16}$$

This property is usually called semi-group property.

The anti-derivative property is also given by

$$\mathbb{D}_q^{\alpha} I_q^{\alpha} f(x) = f(x), \tag{2.17}$$

and

$$I^{\alpha} I^{\beta} f(t) = I^{\beta} I^{\alpha} f(t), \tag{2.18}$$

for all $\alpha, \beta \geq 0$. This is also called the commutative property.

Definition 2.3. Let f be an integrable function on $[a,b]$ and $x \in (a,b)$, then

$$\left(I_q^{\alpha} f\right)(x) = \frac{1}{\Gamma_q(\alpha)} \int_a^x (x - qt)^{(\alpha-1)} f(t) d_q t, \tag{2.19}$$

for all $\alpha \in \mathbb{R}^{+}$.

3 Results and Discussion

We begin this section with a lemma.

Lemma 3.1. *Let f and g be positive and increasing functions on $[0,\infty)$ such that $f(y) > f(x)$ and $g(y) > g(x)$ for $y > x$, then*

$$f(y)g(y) + f(x)g(x) > f(y)g(x) + f(x)g(y) \tag{3.1}$$

holds for all $x, y \in [0, \infty)$.

Proof. Given that $f(y) > f(x)$, then

$$\frac{f(y)}{g(y)} - \frac{f(x)}{g(y)} > 0. \tag{3.2}$$

and for $g(y) > g(x)$, we have

$$\frac{g(y)}{f(y)} - \frac{g(x)}{f(y)} > 0. \tag{3.3}$$

Thus

$$\left(\frac{f(y)}{g(y)} - \frac{f(x)}{g(y)}\right)\left(\frac{g(y)}{f(y)} - \frac{g(x)}{f(y)}\right) > 0. \tag{3.4}$$

Implies

$$f(y)g(y) + f(x)g(x) - f(y)g(x) - f(x)g(y) > 0, \tag{3.5}$$

$$f(y)g(y) + f(x)g(x) > f(y)g(x) + f(x)g(y) \tag{3.6}$$

as required. □

Lemma 3.2. *Let* $\alpha > 0$ *and* $t, x \in \mathbb{R}^+$ *then*

$$\int_0^t \frac{(t-qx)^{\alpha-1}}{\Gamma_q(\alpha)} d_qx = \frac{t^\alpha(1-(1-q)^\alpha)}{[\alpha]_q\Gamma_q(\alpha)} \tag{3.7}$$

holds for $t \neq qx$.

Proof. Considering the LHS of (3.2) we have

$$\begin{aligned} \int_0^t \frac{(t-xq)^{\alpha-1}}{\Gamma_q(\alpha)} d_qx &= \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-xq)^{\alpha-1} d_qx, & (3.8)\\ &= \frac{t^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^t \left(1-\left(\frac{xq}{t}\right)\right)^{\alpha-1} d_qx. & (3.9)\end{aligned}$$

Let $\tau = \frac{xq}{t}$, thus

$$\int_0^t \frac{(t-xq)^{\alpha-1}}{\Gamma_q(\alpha)} d_qx = \frac{t^\alpha}{\Gamma_q(\alpha)}\int_0^q (1-\tau)^{\alpha-1} d_q\tau. \tag{3.10}$$

Also, let $\psi = 1-\tau$, thus

$$\begin{aligned} \int_0^t \frac{(t-xq)^{\alpha-1}}{\Gamma_q(\alpha)} d_qx &= -\frac{t^\alpha}{q\Gamma_q(\alpha)}\int_1^{1-q} (\psi)^{\alpha-1} d_q\psi, & \\ &= \frac{t^\alpha}{q\Gamma_q(\alpha)}\left[\int_0^1 (\psi)^{\alpha-1} d_q\psi - \int_0^{1-q} (\psi)^{\alpha-1} d_q\psi\right]. & (3.11)\\ & & (3.12)\end{aligned}$$

Using (2.7) and $\sum_{n=0}^{\infty} q^{\alpha n} = \frac{1}{1-q^\alpha}$, we have

$$\begin{aligned} \int_0^a \psi^{\alpha-1} d_q\psi = (1-q)a\sum_{n=0}^{\infty}(aq^n)^{\alpha-1}q^n &= \frac{a^\alpha(1-q)}{1-q^\alpha}, & (3.13)\\ &= \frac{a^\alpha}{[\alpha]_q}. & (3.14)\end{aligned}$$

Hence (3.11) becomes

$$\begin{aligned} \int_0^t \frac{(t-qx)^{\alpha-1}}{\Gamma_q(\alpha)} d_qx &= \frac{t^\alpha}{q\Gamma_q(\alpha)}\left[\frac{1^\alpha}{[\alpha]_q} - \frac{(1-q)^\alpha}{[\alpha]_q}\right], & (3.15)\\ &= \frac{t^\alpha(1-(1-q)^\alpha)}{[\alpha]_q q\Gamma_q(\alpha)} & (3.16)\end{aligned}$$

as requried. □

Lemma 3.3. *Let f and g be positive and increasing functions on $\mathbb{R}^+$ such that $f(y) > f(x)$ and $g(y) > g(x)$ for $y > x$, then*

$$\left(I_q^\alpha fg\right)(t) + \frac{t^\alpha(1-(1-q)^\alpha)}{[\alpha]_q q\Gamma_q(\alpha)} fg(\eta) > I_q^\alpha f(t)g(\eta) + f(\eta)I_q^\alpha g(t) \tag{3.17}$$

for $\eta, t, \alpha \neq 0$.

Proof. Using inequality (3.1) we have

$$f(\xi)g(\xi) + f(\eta)g(\eta) > f(\xi)g(\eta) + f(\eta)g(\xi). \tag{3.18}$$

Multiply (3.18) by $\frac{(t-\xi q)^{\alpha-1}}{\Gamma_q(\alpha)}$ for $t > \xi q$ and integrating yields

$$\begin{gathered}\int_0^t \frac{(t-\xi q)^{\alpha-1}}{\Gamma_q(\alpha)} fg(\xi)d_q\xi + \int_0^t \frac{(t-\xi q)^{\alpha-1}}{\Gamma_q(\alpha)} fg(\eta)d_q\xi > \\ \int_0^t \frac{(t-\xi q)^{\alpha-1}}{\Gamma_q(\alpha)} f(\xi)g(\eta)d_q\xi + \int_0^t \frac{(t-\xi q)^{\alpha-1}}{\Gamma_q(\alpha)} f(\eta)g(\xi)d_q\xi.\end{gathered} \tag{3.19}$$

Applying Definition 2.3 and Lemma 3.2 we obtain

$$\left(I_q^\alpha fg\right)(t) + \frac{t^\alpha(1-(1-q)^\alpha)}{[\alpha]_q q\Gamma_q(\alpha)} fg(\eta) > I_q^\alpha f(t)g(\eta) + f(\eta)I_q^\alpha g(t) \tag{3.20}$$

as requried. □

Theorem 3.4. *Let f and g be positive and increasing functions on $[0, t]$, then*

$$\left(I_q^\alpha fg\right)(t) > \frac{q\Gamma_q(\alpha)[\alpha]_q}{t^\alpha(1-(1-q)^\alpha)} \left(I_q^\alpha f\right)(t)\left(I_q^\alpha g\right)(t) \tag{3.21}$$

for $t \in \mathbb{R}^+$ and $\alpha > 0$.

Proof. Let $t > \eta q$ and multiplying inequality (3.20) by $\int_0^t \frac{(t-\eta q)^{\alpha-1}}{\Gamma_q(\alpha)}$, we have

$$\begin{gathered}\int_0^t \frac{(t-\eta q)^{\alpha-1}}{\Gamma_q(\alpha)} I_q^\alpha fg(t)d_q\eta + \int_0^t \left(\frac{(t-\eta q)^{\alpha-1}}{\Gamma_q(\alpha)}\right)\left(\frac{t^\alpha(1-(1-q)^\alpha)}{[\alpha]_q\Gamma_q(\alpha)}\right) fg(\eta)d_q\eta > \\ \int_0^t \frac{(t-\eta q)^{\alpha-1}}{\Gamma_q(\alpha)} f(\xi)g(\eta)d_q\eta + \int_0^t \frac{(t-\eta q)^{\alpha-1}}{\Gamma_q(\alpha)} f(\eta)g(\xi)d_q\eta.\end{gathered} \tag{3.22}$$

Also applying Definition 2.3 and Lemma 3.2 we obtain

$$\frac{t^\alpha(1-(1-q)^\alpha)}{[\alpha]_q q\Gamma_q(\alpha)}\left(I_q^\alpha fg(t)\right) + \frac{t^\alpha(1-(1-q)^\alpha)}{[\alpha]_q q\Gamma_q(\alpha)}\left(I_q^\alpha fg(t)\right) > I_q^\alpha\left(f(t)g(t)\right) + I_q^\alpha\left(f(t)g(t)\right). \tag{3.23}$$

This simplifies to

$$\left(I_q^\alpha fg\right)(t) > \frac{q\Gamma_q(\alpha)[\alpha]_q}{t^\alpha(1-(1-q)^\alpha)} \left(I_q^\alpha f\right)(t)\left(I_q^\alpha g\right)(t) \tag{3.24}$$

as required. □

Corollary 3.5. *Let f and g be integrable functions on $(0, t)$, $t \in \mathbb{R}^+$ and $f(y) > f(x)$ and $g(y) > g(x)$, then*

$$(I_q fg)(t) > \frac{1}{t^\alpha}(I_q f)(t)(I_q g)(t), \tag{3.25}$$

for $y > x$ and $t \neq 0$.

Proof. Put $\alpha = 1$, and $q = 1$ into (3.24) yields the required Corollary. □

Lemma 3.6. *Let f and g be positive and increasing functions on $[0, \infty)$, then*

$$\int_0^t \frac{(t^\alpha - (\eta q)^\alpha)^{\beta-1}(\eta q)^{\alpha-1}}{\alpha^{\beta-1}} d_q\eta = \frac{\alpha^{1-\beta}t^{\alpha\beta}(1-(1-q^\alpha)^\beta)}{q^\alpha[\alpha]_q[\beta]_q} \tag{3.26}$$

for all $t, \alpha, \beta > 0$.

Proof. Considering the LHS of (3.6), we have

$$\begin{aligned}\int_0^t \frac{(t^\alpha - (\eta q)^\alpha)^{\beta-1}(\eta q)^{\alpha-1}}{\alpha^{\beta-1}} d_q\eta &= \int_0^t \alpha^{1-\beta}(t^\alpha - (\eta q)^\alpha)^{\beta-1}(\eta q)^{\alpha-1} d_q\eta, & (3.27)\\ &= \alpha^{1-\beta}t^{\alpha(\beta-1)}\int_0^t (\eta q)^{\alpha-1}\left(1-\left(\frac{\eta q}{t}\right)^\alpha\right)^{\beta-1} d_q\eta. & (3.28)\end{aligned}$$

Let $\tau = \left(\frac{\eta q}{t}\right)^\alpha$, this implies

$$\begin{aligned}\int_0^t \frac{(t^\alpha - (\eta q)^\alpha)^{\beta-1}(\eta q)^{\alpha-1}}{\alpha^{\beta-1}} d_q\eta &= \alpha^{1-\beta}t^{\alpha(\beta-1)}\int_0^t (\eta q)^{\alpha-1}\left(1-\left(\frac{\eta q}{t}\right)^\alpha\right)^{\beta-1}\frac{t^\alpha}{q^\alpha[\alpha]_q(\eta q)^{\alpha-1}} d_q\tau & (3.29)\\ &= \frac{\alpha^{1-\beta}t^{\alpha\beta}}{q^\alpha[\alpha]_q}\int_0^{q^\alpha}(1-\tau)^{\beta-1} d_q\tau. & (3.30)\end{aligned}$$

Again let $\psi = 1 - \tau$, then

$$\begin{aligned}\int_0^t \frac{(t^\alpha - \eta^\alpha)^{\beta-1}\eta^{\alpha-1}}{\alpha^{\beta-1}} d_q\eta &= -\frac{\alpha^{1-\beta}t^{\alpha\beta}}{q^\alpha[\alpha]_q}\int_1^{1-q^\alpha}(\psi)^{\beta-1} d_q\psi.\\ &= \frac{\alpha^{1-\beta}t^{\alpha\beta}}{q^\alpha[\alpha]_q}\left[\int_0^1(\psi)^{\beta-1} d_q\psi - \int_0^{1-q^\alpha}(\psi)^{\beta-1} d_q\psi\right].\end{aligned}$$

Thus

$$\begin{aligned}\int_0^t \frac{(t^\alpha - \eta^\alpha)^{\beta-1}\eta^{\alpha-1}}{\alpha^{\beta-1}} d_q\eta &= \frac{\alpha^{1-\beta}t^{\alpha\beta}}{q^\alpha[\alpha]_q}\left[\frac{1^\beta}{[\beta]_q} - \frac{(1-q^\alpha)^\beta}{[\alpha]_q}\right],\\ &= \frac{\alpha^{1-\beta}t^{\alpha\beta}(1-(1-q^\alpha)^\beta)}{q^\alpha[\alpha]_q[\beta]_q}. & (3.31)\end{aligned}$$

□

Theorem 3.7. *Let f and g be positive and increasing functions on $[0, \infty)$. Then*

$$I_q^{\alpha+\beta} fg(t) > \frac{q\Gamma_q(\beta)\Gamma_q(\alpha)[\alpha]_q}{\omega\Gamma_q(\alpha)[\alpha]_q + t^\alpha\Gamma_q(\beta)(1-(1-q)^\alpha)} I_q^{\alpha+\beta} f(t)g(t) \tag{3.32}$$

for all $t, \alpha, \beta > 0$.

Proof. By inequality (3.20) we have

$$\int_0^t \left(I_q^\alpha fg\right)(t)\frac{(t^\alpha-\eta^\alpha)^{\beta-1}\eta^{\alpha-1}}{\Gamma_q(\beta)\alpha^{\beta-1}}d_q\eta + \int_0^t \frac{t^\alpha(1-(1-q)^\alpha)}{[\alpha]_q\Gamma_q(\alpha)}I_q^\alpha g(t)\frac{(t^\alpha-\eta^\alpha)^{\beta-1}\eta^{\alpha-1}}{\Gamma_q(\beta)\alpha^{\beta-1}}d_q\eta > \int_0^t I_q^\alpha f(t)g(\eta)\frac{(t^\alpha-\eta^\alpha)^{\beta-1}\eta^{\alpha-1}}{\Gamma_q(\beta)\alpha^{\beta-1}}d_q\eta + \int_0^t f(\eta)I_q^\alpha g(t)\frac{(t^\alpha-\eta^\alpha)^{\beta-1}\eta^{\alpha-1}}{\Gamma_q(\beta)\alpha^{\beta-1}}d_q\eta \quad (3.33)$$

for $t > \eta$.

Applying Definition 2.3 and Lemma 3.6 we obtain

$$\frac{\alpha^{1-\beta}t^{\alpha\beta}(1-(1-q^\alpha)^\beta)}{q^\alpha\Gamma_q(\beta)[\alpha]_q[\beta]_q}(I_q^\alpha)I_q^\beta fg(t) + \frac{t^\alpha(1-(1-q)^\alpha)}{q[\alpha]_q\Gamma_q(\alpha)}(I_q^\alpha)I_q^\beta fg(t) > I_q^\alpha f(t)I_q^\beta g(t) + I_q^\beta f(t)I_q^\alpha g(t). \quad (3.34)$$

Let $\omega = \frac{\alpha^{1-\beta}t^{\alpha\beta}(1-(1-q^\beta)^\alpha)}{q^\alpha[\alpha]_q[\beta]_q}$ and substitute into (3.34) yields

$$\frac{\omega}{\Gamma_q(\beta)}I_q^\alpha I_q^\beta fg(t) + \frac{t^\alpha(1-(1-q)^\alpha)}{q[\alpha]_q\Gamma_q(\alpha)}I_q^\alpha I_q^\beta fg(t) > I_q^\alpha f(t)I_q^\beta g(t) + I_q^\beta f(t)I_q^\alpha g(t). \quad (3.35)$$

Applying (2.16) yields

$$\frac{\omega}{\Gamma_q(\beta)}I_q^{\alpha+\beta} fg(t) + \frac{t^\alpha(1-(1-q)^\alpha)}{q[\alpha]_q\Gamma_q(\alpha)}I_q^{\alpha+\beta} fg(t) > I_q^{\alpha+\beta} f(t)g(t) + I^{\alpha+\beta} f(t)g(t). \quad (3.36)$$

Thus

$$I_q^{\alpha+\beta} fg(t) > \frac{q\Gamma_q(\beta)\Gamma_q(\alpha)[\alpha]_q}{\omega\Gamma_q(\alpha)[\alpha]_q + t^\alpha\Gamma_q(\beta)(1-(1-q)^\alpha)}I_q^{\alpha+\beta} f(t)g(t) \quad (3.37)$$

as required. □

Corollary 3.8. *Let f and g be positive and increasing functions on $[0,\infty)$, then*

$$I^{\alpha+1} fg(t) > \frac{1}{t^\alpha}I^{\alpha+1} f(t)g(t) \quad (3.38)$$

is valid for $t \neq 0$.

Proof. By leting $\omega = 0$, $\beta = 1, q = 1$ and substituting into (3.37) yields the result. □

Remark 3.1. Let $\alpha = 1$ in (3.38), then

$$I^2 fg(t) > \frac{1}{t}I^2 f(t)g(t) \quad (3.39)$$

for $t \neq 0$.

4 Conclusions

This study presented some new inequalities involving Chebyshev and Riemann-Liouville fractional integral inequalities for positive and increasing functions using q-Calculus. The results will contribute significantly to knowledge that exist in this field of study and the academic world.

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Competing Interests

Authors have declared that no competing interests exist.

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