Common Fixed Point Theorems for Compatible Maps in Generalized Fuzzy Metric Spaces

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Authors' contributions

This work was carried out in collaboration among all authors and equally contributed to this article. All authors read and approved the final manuscript.

Original Research Article

Abstract

In this paper, we consider generalized fuzzy metric spaces and provide existence and uniqueness fixed point results. First, we use compatible maps of type (β) to prove fixed point results, then we introduce weakly compatible maps to approximate common fixed point results by using an implicit relation.

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1   Introduction

Fuzzy set initiated by Zadeh [1] is considered to be one of the most powerful tools in the areas of artificial intelligence, computer science, control engineering, decision theory, robotics, etc. as it can deal with the problems of uncertainty in complex systems. In applications of fuzzy set theory, the field of engineering has undoubtedly been a leader. In 1975, Kramosil and Michalek [2] introduced fuzzy metric space as a generalization of a metric space. In 1994, George and Veeramani [3] modified the notion of fuzzy metric spaces by using continuous t-norms which paves a way to establish many fixed point results in fuzzy metric spaces. In 2006, the concept of M-fuzzy metric space was introduced by Sedghi and Shobe [4] and proved a common fixed point theorems in it. Veerapandi et al. [5] defined generalized M-fuzzy metric space and established some fixed point and coincident point theorems. Mishra et al. [6] formulated the notion of weakly compatible mappings in fuzzy settings and proved some fixed point theorems on fuzzy metric space.

In 2007, Pant et al. [7] studied the common fixed points of a pair of non-compatible maps in fuzzy metric spaces. Cho et al. [8] defined the notion called compatible maps of type (β). In 2008, Altun and Turkoglu [9] developed the notion of compatible maps of type (β) on complete fuzzy metric space and proved common fixed point theorems with the help of an implicit relation.

In this paper, we first formulate the definition of compatible maps of type (β) in generalized fuzzy metric spaces. Thereafter, we obtain common fixed point theorems for compatible maps of type (β) and weakly compatible which generalize, extend, unify and fuzzify several well-known fixed point theorems.

2   Preliminaries

Definition 2.1. [10]   A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous t - norm if it satisfies the following conditions:

    (i) $*$ is associative and commutative,
    (ii) $*$ is continuous,
    (iii) $a * 1 = a$, for all $a \in [0,1]$,
    (iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$.
Examples for continuous t - norm are $a * b = \min\{a, b\}$ and $a * b = ab$.

Definition 2.2. [5]   A 3 - tuple $(X, \mathcal{M}, *)$ is called generalized fuzzy metric space if X is an arbitrary non - empty set, $*$ is a continuous t - norm, and M is a fuzzy set on $X^3 \times (0, \infty)$ satisfying the following conditions; for each $x, y, z, a \in X$ and $t, s > 0$,
(GFM - 1) $M(x, y, z, t) > 0$,
(GFM - 2) $M(x, y, z, t) = 1$ if $x = y = z$,
(GFM - 3) $M(x, y, z, t) = M(p\{x, y, z\}, t)$, where p is a permutation function,
(GFM - 4) $M(x, y, a, t) * M(a, z, z, s) \leq M(x, y, z, t + s)$,
(GFM - 5) $M(x, y, z, .) : (0, \infty) \to [0,1]$ is continuous,
(GFM - 6) $\lim\limits_{t \to \infty} M(x, y, z, t) = 1$.

Definition 2.3. [11] A pair (A, S) of self-mappings of a generalized fuzzy metric space is said to be weak compatible or coincidentally commuting if A and S commute at their coincidence points, i.e. for $x \in X$ if $Ax = Sx$ then $ASx = SAx$.

Definition 2.4. Let A and B be maps from an generalized fuzzy metric space $(X, M, *)$ into itself. The maps A and B are said to be compatible of type (β) if $\lim\limits_{n \to \infty} M(AAx_n, SSx_n, SSx_n, t) = 1$, for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim\limits_{n \to \infty} Ax_n = \lim\limits_{n \to \infty} Bx_n = z$ for some $z \in X$.

Definition 2.5. Let $I = [0,1]$, $*$ be a continuous t - norm and F be the set of all real continuous functions $F : I^6 \to \mathbb{R}$ satisfying the following conditions:

(i) F is no increasing in the fifth and sixth variables,

(ii) If, for some constant $k \in (0,1)$ we have

    (a). $F\left(u(kt), v(t), v(t), u(t), 1, u\left(\frac{1}{2}\right) * v\left(\frac{t}{2}\right)\right) \geq 1$ or

    (b). $F\left(u(kt), v(t), u(t), v(t), u\left(\frac{t}{2}\right) * v\left(\frac{t}{2}\right), 1\right) \geq 1$.

for any fixed $t > 0$ and any non - decreasing functions $u, v : (0, \infty) \to I$ with $0 \leq u(t), v(t) \leq 1$ then there exists $h \in (0,1)$ with $u(ht) \geq v(t) * u(t)$, if for some constant $k \in (0,1)$ we have $F(u(kt), u(t), 1, 1, u(t), u(t)) \geq 1$, for any fixed $t > 0$ and any non-decreasing function $u : (0, \infty) \to I$ then $u(kt) \geq u(t)$.

Lemma 2.1. *In a generalized fuzzy metric space $(X, M, *)$ limit of a sequence is unique.*

Proof. Let $\{x_n\}$ be a sequence in generalized fuzzy metric space X and suppose $x_n \to x$ and $x_n \to y$ for some $x, y \in X$. We shall show that $x = y$.
We have $M(x, x, y, t+s) \geq M(x, x, x_n, t) * M(x_n, y, y, s)$ and let $n \to \infty$.
Then $M(x, x, y, t+s) \geq \lim_{n \to \infty} M(x, x, x_n, t) * \lim_{n \to \infty} M(x_n, y, y, s) = 1 * 1 = 1$.
Thus $M(x, x, y, t+s) = 1$, for all $t, s > 0$. So, $x = y$. $\qquad\qquad\square$

Lemma 2.2. *[5] Let $(X, M, *)$ be a generalized fuzzy metric space. Then*

  (i) *For all $x, y, z \in X, M(x, y, z.)$ is a non decreasing function.*

  (ii) *If there exists $k \in (0,1)$ such that for all $x, y, z \in X, M(x, y, z, kt) \geq M(x, y, z, t)$ for all $t > 0$, then $x = y = z$.*

  (iii) *If there exists a number $k \in (0,1)$ such that $M(x_{n+2} x_{n+1}, x_{n+1}, kt) \geq M(x_{n+1}, x_n, x_n, t)$ for all $t > 0$ and $n \in \mathbb{N}$.*

Then $\{x_n\}$ is a Cauchy sequence in X.

3 Main Results

In this section, we prove a common fixed point theorems for compatible map of type (β) in generalized fuzzy metric spaces .

Theorem 3.1. *Let $(X, M, *)$ be a complete generalized fuzzy metric space and A, B, G, H, S, T, P, Q and R be mappings from X into itself such that the following conditions are satisfied:*

(3.1.1) $P(X) \subset ST(X), Q(X) \subset AB(X)$ and $R(X) \subset GH(X)$,

(3.1.2) (P, AB) is compatible of type (β) and $(Q, GH), (R, ST)$ are weak compatible,

(3.1.3) There exists $k \in (0,1)$ such that for every $x, y, z \in X$ and $t > 0$

$$F\left(\begin{array}{c} M^2(Px, Qy, Rz, kt), M^2(ABx, STz, GHy, t) \\ M^2(Px, STz, GHy, t), M^2(Rz, Qy, STz, t) \\ M^2(GHy, Px, ABx, t), M^2(ABxQy, Rz, t) \end{array}\right) \geq 1.$$

Then A, B, G, H, S, T, P, Q and R have a unique common fixed point in X.

Proof. Let $x_0 \in X$, then from (3.1.1) we have $x_1, x_2, x_3 \in X$ such that $Px_0 = STx_1, Qx_1 = ABx_2$ and $Rx_2 = GHx_3$.

Inductively, we construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $n \in \mathbb{N}$.

$Px_{2n-2} = STx_{2n-1} = y_{2n-1}, Qx_{2n-1} = ABx_{2n} = y_{2n}$ and $Rx_{2n} = GHx_{2n+1} = y_{2n+1}$.

Put $x = x_{2n}, y = x_{2n+1}$ and $z = x_{2n+1}$ in (3.1.2) then we have

$$F\left(\begin{array}{c} M^2(Px_{2n}, Qx_{2n+1}, Rx_{2n+1}, kt), M^2(ABx_{2n}, STx_{2n+1}, GHx_{2n+1}, t) \\ M^2(Px_{2n}, STx_{2n+1}, GHx_{2n+1}, t), M^2(Rx_{2n+1}, Qx_{2n+1}, STx_{2n+1}, t) \\ M^2(GHx_{2n+1}, Px_{2n}, ABx_{2n}, t), M^2(ABx_{2n}, Qx_{2n+1}, Rx_{2n+1}, t) \end{array}\right) > 1$$

$$F\left(\begin{array}{c} M^2(y_{2n+1}, y_{2n+2}, y_{2n+2}, kt), M^2(y_{2n}, y_{2n+1}, y_{2n+1}, t) \\ M^2(y_{2n+1}, y_{2n+1}, y_{2n+1}, t), M^2(y_{2n+2}, y_{2n+2}, y_{2n+1}, t) \\ M^2(y_{2n+1}, y_{2n+1}, y_{2n}, t), M^2(y_{2n}, y_{2n+2}, y_{2n+2}, t) \end{array}\right) > 1$$

$$F\left(\begin{array}{c} M^2(y_{2n+1}, y_{2n+2}, y_{2n+2}, kt), M^2(y_{2n}, y_{2n+1}, y_{2n+1}, t) \\ M^2(y_{2n+1}, y_{2n+1}, y_{2n+1}, t), M^2(y_{2n+2}, y_{2n+2}, y_{2n+1}, t) \\ M^2(y_{2n+1}, y_{2n+1}, y_{2n}, t), M^2\left(y_{2n}, y_{2n+1}, y_{2n+1}, \frac{t}{2}\right) \\ *M^2\left(y_{2n+1}, y_{2n+2}, y_{2n+2}, \frac{t}{2}\right) \end{array}\right) > 1$$

From condition (3.1.1) we have

$$M^2(y_{2n+1}, y_{2n+2}, y_{2n+2}, kt) \geq M^2\left(y_{2n}, y_{2n+1}, y_{2n+1}, \frac{t}{2}\right) * M^2\left(y_{2n+1}, y_{2n+2}, y_{2n+2}, \frac{t}{2}\right)$$

We have

$$M^2(y_{2n+1}, y_{2n+2}, y_{2n+2}, kt) \geq M^2\left(y_{2n}, y_{2n+1}, y_{2n+1}, \frac{t}{2}\right).$$

That is $M(y_{2n+1}, y_{2n+2}, y_{2n+2}, kt) \geq M\left(y_{2n}, y_{2n+1}, y_{2n+1}, \frac{t}{2}\right)$.

Similarly, we have $M(y_{2n+2}, y_{(}2n + 3), y_{(}2n + 3), kt) \geq M\left(y_{2n+1}, y_{2n+2}, y_{2n+2}, \frac{t}{2}\right)$.

Thus, we have $M(y_{n+1}, y_{n+2}, y_{n+2}, kt) \geq M\left(y_n, y_{n+1}, y_{n+1}, \frac{t}{2}\right)$.

$M(y_{n+1}, y_{n+2}, y_{n+2}, t) \geq M\left(y_n, y_{n+1}, y_{n+1}, \frac{t}{2k}\right)$.

$M(y_n, y_{n+1}, y_{n+1}, t) \geq M\left(y_0, y_1, y_1, \frac{t}{2^{nk}}\right) \to 1$ as $n \to \infty$ and hence $M(y_n, y_{n+1}, y_{n+1}, t) \to 1$ as $n \to \infty$ for all $t > 0$ for each $\epsilon > 0$ and $t > 0$, we can choose $n_0 \in \mathbb{N}$ such that

$M(y_n, y_{n+1}, y_{n+1}, t) > 1 - \epsilon$ for all $n > n_0$. For any $m, n \in \mathbb{N}$ we suppose that $m \geq n$. Then we have

$$M(y_n, y_m, y_m, t) \geq \left\{\begin{array}{c} M\left(y_n, y_{n+1}, y_{n+1}, \frac{t}{m-n}\right) * \\ M\left(y_{n+1}, y_{n+2}, y_{n+2}, \frac{t}{m-n}\right) * \cdots * \\ M\left(y_{m-1}, y_m, y_m, \frac{t}{m-n}\right). \end{array}\right\}$$

$$M(y_n, y_m, y_m, t) \geq (1 - \epsilon) * (1 - \epsilon) * \cdots * (1 - \epsilon)(m - n) \text{ times}$$

$$M(y_n, y_m, y_m, t) \geq (1 - \epsilon)$$

and hence $\{y_n\}$ is a Cauchy sequence in X.

Since $(X, M, *)$ is complete, $\{y_n\}$ converges to some point $w \in X$.

Also its subsequences converges to the same point $w \in X$. That is

$$\{Px_{2n+2}\} \to w \text{ and } \{STx_{2n+1}\} \to w, \tag{3.1}$$

$$\{Qx_{2n+1}\} \to w \text{ and } \{GHx_{2n}\} \to w, \tag{3.2}$$

$$\{Rx_{2n}\} \to w \text{ and } \{ABx_{2n-1}\} \to w. \tag{3.3}$$

As (P, AB) is compatible pair of type (β), we have
$M(PPx_{2n}, (AB)(AB)x_{2n}, (AB)(AB)x_{2n}, t) = 1$, for all $t > 0$ Or $M(PPx_{2n}, ABw, ABw, t) = 1$.

Therefore, $PPx_{2n} \to ABw$.

Put $x = (AB)x_{2n}, y = x_{2n+1}$ and $z = x_{2n+1}$ in (3.1.3) we have

$$F \begin{pmatrix} M^2(P(AB)x_{2n}, Qx_{2n+1}, Rx_{2n+1}, kt), M^2(AB(AB)x_{2n}, STx_{2n+1}, GHx_{2n+1}, t) \\ M^2(P(AB)x_{2n}, STx_{2n+1}, GHx_{2n+1}, t), M^2(Rx_{2n+1}, Qx_{2n+1}, STx_{2n+1}, t) \\ M^2(GHx_{2n+1}, P(AB)x_{2n}, AB(AB)x_{2n}, t), M^2(AB(AB)x_{2n}, Qx_{2n+1}, Rx_{2n+1}, t) \end{pmatrix} > 1$$

Taking $n \to \infty$ and (3.1.1) we get $M^2((AB)w, w, w, kt) \geq M^2((AB)w, w, w, t)$.

That is $M((AB)w, w, w, kt) \geq M((AB)w, w, w, t)$.

Therefore we have,

$$ABw = w \tag{3.4}$$

Put $x = w, y = x_{2n+1}$ and $z = x_{2n+1}$ in (3.1.3) we have

$$F \begin{pmatrix} M^2(Pw, Qx_{2n+1}, Rx_{2n+1}, kt), M^2(ABw, STx_{2n+1}, GHx_{2n+1}, t) \\ M^2(Pw, STx_{2n+1}, GHx_{2n+1}, t), M^2(Rx_{2n+1}, Qx_{2n+1}, STx_{2n+1}, t) \\ M^2(GHx_{2n+1}, Pw, ABw, t), M^2(ABw, Qx_{2n+1}, Rx_{2n+1}, t) \end{pmatrix} \geq 1$$

Taking $n \to \infty$ and (3.1.1) we get, $M^2(Pw, w, w, kt) \geq M^2(Pw, w, w, t)$.

That is $M(Pw, w, w, kt) \geq M(Pw, w, w, t)$.

Therefore we have by using lemma (2.1), we get $Pw = w$. So, we have $ABw = Pw = w$.

Putting $x = Bw, y = x_{2n+1}$ and $z = x_{2n+1}$ in (3.1.3) we have

$$F \begin{pmatrix} M^2(PBw, Qx_{2n+1}, Rx_{2n+1}, kt), M^2(ABBw, STx_{2n+1}, GHx_{2n+1}, t) \\ M^2(PBw, STx_{2n+1}, GHx_{2n+1}, t), M^2(Rx_{2n+1}, Qx_{2n+1}, STx_{2n+1}, t) \\ M^2(GHx_{2n+1}, PBw, ABBw, t), M^2(ABBw, Qx_{2n+1}, Rx_{2n+1}, t) \end{pmatrix} > 1.$$

Taking $n \to \infty$ (3.1.1) and using (3.1.4), we get

$$M^2(Bw, w, w, kt) \geq M^2(Bw, w, w, t).$$

That is $M(Bw, w, w, kt) \geq M(Bw, w, w, t)$.

Therefore by using lemma (2.1), we have $Bw = w$, and also we have $ABw = w$ implies $Aw = w$. Therefore

$$Aw = Bw = Pw = w. \tag{3.5}$$

As $P(X) \subset ST(X)$, there exists $u \in X$ such that $w = Pw = STu$.

Putting $x = x_{2n}, y = x_{2n}$ and $z = u$ in (3.1.3) we have

$$F \begin{pmatrix} M^2(Px_{2n}, Qx_{2n}, Ru, kt), M^2(ABx_{2n}, STu, GHx_{2n}, t) \\ M^2(Px_{2n}, STu, GHx_{2n}, t), M^2(Ru, Qx_{2n}, STu, t) \\ M^2(GHx_{2n}, Px_{2n}, ABx_{2n}, t), M^2(ABx_{2n}, Qx_{2n}, Ru, t) \end{pmatrix} > 1.$$

Taking $n \to \infty$ and using in (3.1.4) and (3.1.5) we get

$$F \begin{pmatrix} M^2(w, w, Ru, kt), M^2(w, STu, w, t) \\ M^2(w, STu, w, t), M^2(Ru, w, STu, t) \\ M^2(w, w, w, t), M^2(w, w, Ru, t) \end{pmatrix} > 1.$$

$M^2(w, w, Ru, kt) \geq M^2(w, w, Ru, t)$.

That is $M(w, w, Ru, kt) \geq M(w, w, Ru, t)$, we have $Ru = w$. Hence $STu = w = Ru$.

Hence (R, ST) is weak compatible, therefore, we have $RSTu = STRu$. Thus $Rw = STw$.

Putting $x = x_{2n}, y = x_{2n}$ and $z = w$ in (3.1.3) we get

$$F \begin{pmatrix} M^2(Px_{2n}, Qx_{2n}, Rw, kt), M^2(ABx_{2n}, STw, GHx_{2n}, t) \\ M^2(Px_{2n}, STw, GHx_{2n}, t), M^2(Rw, Qx_{2n}, STw, t) \\ M^2(GHx_{2n}, Px_{2n}, ABx_{2n}, t), M^2(ABx_{2n}, Qx_{2n}, Rw, t) \end{pmatrix} > 1.$$

Taking $n \to \infty$ and using (3.1.5) we get

$$
F \left(
\begin{array}{c}
M^2(w, w, Rw, kt), M^2(w, STw, w, t) \\
M^2(w, STw, w, t), M^2(Rw, w, w, t) \\
M^2(w, w, w, t), M^2(w, w, Rw, t)
\end{array}
\right) > 1.
$$

$M^2(w, w, Ru, kt) \geq M^2(w, w, Ru, t)$ and hence $M(w, w, Ru, kt) \geq M(w, w, Ru, t)$, we get $Rw = w$.
Putting $x = x_{2n}, y = x_{2n}$ and $z = Tw$ in (3.1.3) we get

$$
F \left(
\begin{array}{c}
M^2(Px_{2n}, Qx_{2n}, RTw, kt), M^2(ABx_{2n}, STTw, GHx_{2n}, t) \\
M^2(Px_{2n}, STTw, GHx_{2n}, t), M^2(RTw, Qx_{2n}, STTw, t) \\
M^2(GHx_{2n}, Px_{2n}, ABx_{2n}, t), M^2(ABx_{2n}Qx_{2n}, RTw, t)
\end{array}
\right) > 1.
$$

As $RT = TR$ and $ST = TS$ we have $RTw = TRw = T$ and $ST(Tw) = T(STw) = TRw = Tw$.
Taking $n \to \infty$ we get

$$
F \left(
\begin{array}{c}
M^2(w, w, Tw, kt), M^2(w, Tw, w, t) \\
M^2(w, Tw, w, t), M^2(Tw, w, Tw, t) \\
M^2(w, w, w, t), M^2(w, w, Tw, t)
\end{array}
\right) > 1
$$

and

$$
M^2(w, w, Tw, kt) \geq M^2(w, w, Tw, t).
$$

Therefore $M(w, w, Tw, kt) \geq M(w, w, Tw, t)$
Therefore by lemma (2.1), we have $Tw = w$.
Now, $STw = Tw = w$ implies $Sw = w$. Hence

$$
Sw = Tw = Rw = w. \tag{3.6}
$$

As $R(X) \subset GH(X)$, there exists $u \in X$ such that $w = Rw = GHu$.
Putting $x = x_{2n}, y = u$ and $z = x_{2n}$ in (3.1.3) we get

$$
F \left(
\begin{array}{c}
M^2(Px_{2n}, Qu, Rx_{2n}, kt), M^2(ABx_{2n}, STx_{2n}, GHu, t) \\
M^2(Px_{2n}, STx_{2n}, GHu, t), M^2(Rx_{2n}, Qu, STx_{2n}, t) \\
M^2(GHu, Px_{2n}, ABx_{2n}, t), M^2(ABx_{2n}, Qu, Rx_{2n}, t)
\end{array}
\right) > 1.
$$

Taking $n \to \infty$ and using in (3.1.4) and (3.1.5) we get

$$
F \left(
\begin{array}{c}
M^2(w, Qu, w, kt), M^2(w, w, GHu, t) \\
M^2(w, w, GHu, t), M^2(w, Qu, w, t) \\
M^2(GHu, w, w, t), M^2(w, Qu, w, t)
\end{array}
\right) > 1.
$$

$$
M^2(w, Qu, w, kt) \geq M^2(w, Qu, w, t).
$$

That is $M(w, Qu, w, kt) \geq M(w, Qu, w, t)$, we have $Qu = w$. Hence $GHu = w = Qu$.
Hence (Q, GH) is weak compatible, therefore, we have $QGHu = GHQu$. Thus $Qw = GHw$.
Putting $x = x_{2n}, y = w$ and $z = x_{2n}$ in (3.1.3) we get

$$
F \left(
\begin{array}{c}
M^2(Px_{2n}, Qw, Rx_{2n}, kt), M^2(ABx_{2n}, STx_{2n}, GHw, t) \\
M^2(Px_{2n}, STx_{2n}, GHw, t), M^2(Rx_{2n}, Qw, STx_{2n}, t) \\
M^2(GHw, Px_{2n}, ABx_{2n}, t), M^2(ABx_{2n}, Qw, Rx_{2n}, t)
\end{array}
\right) > 1.
$$

Taking $n \to \infty$ and using (3.1.5) we get

$$
F \left(
\begin{array}{c}
M^2(w, Qw, w, kt), M^2(w, w, GHw, t) \\
M^2(w, w, GHw, t), M^2(w, Qw, w, t) \\
M^2(GHw, w, w, t), M^2(w, Qw, w, t)
\end{array}
\right) > 1.
$$

$M^2(w, Qw, w, kt) \geq M^2(w, Qw, w, t)$ and hence $M(w, Qw, w, kt) \geq M(w, Qw, w, t)$, we get $Qw = w$.
Putting $x = x_{2n}, y = Hw$ and $z = x_{2n}$ in (3.1.3) we get

$$
F \left(\begin{array}{c}
M^2(Px_{2n}, QHw, Rx_{2n}, kt), M^2(ABx_{2n}, STx_{2n}, GHHw, t) \\
M^2(Px_{2n}, STx_{2n}, GHHw, t), M^2(Rx_{2n}, QHw, STx_{2n}, t) \\
M^2(GHHw, Px_{2n}, ABx_{2n}, t), M^2(ABx_{2n}QHw, Rx_{2n}, t)
\end{array} \right) > 1.
$$

As $QH = HQ and GH = HG$, we have
$QHw = HQw = H$ and $GH(Hw) = H(GHw) = HQw = Hw$.
Taking $n \to \infty$ we get

$$
F \left(\begin{array}{c}
M^2(w, Hw, w, kt), M^2(w, w, Hw, t) \\
M^2(w, w, Hw, t), M^2(w, Hw, w, t) \\
M^2(Hw, w, w, t), M^2(w, Hw, w, t)
\end{array} \right) > 1
$$

and

$$
M^2(w, Hw, w, kt) \geq M^2(w, Hw, w, t).
$$

Therefore $M(w, Hw, w, kt) \geq M(w, Hw, w, t)$. Therefore by lemma (2.1), we have $Hw = w$.
Now, $GHw = Hw = w$ implies $Gw = w$. Hence

$$
Gw = Hw = Qw = w. \tag{3.7}
$$

Combining (3.6) and (3.7) we have

$$
Aw = Bw = Rw = Pw = Sw = Tw = Qw = Gw = Hw = w.
$$

Hence w is a common fixed point of A, B, G, H, S, T, P, Q and R.
Uniqueness: Let u be another common fixed point of A, B, G, H, S, T, P, Q and R.
Then $Au = Bu = Ru = Pu = Su = Tu = Qu = Gu = Hu = u$.
Putting $x = u, y = w$ and $z = w$ in (3.1.3) we get

$$
F \left(\begin{array}{c}
M^2(Pu, Qw, Rw, kt), M^2(ABu, STw, GHw, t) \\
M^2(Pu, STw, GHw, t), M^2(Rw, Qw, STw, t) \\
M^2(GHw, Pu, ABu, t), M^2(ABu, Qw, Rw, t)
\end{array} \right) > 1.
$$

Taking limit both side then we get

$$
F \left(\begin{array}{c}
M^2(u, w, w, kt), M^2(u, w, w, t) \\
M^2(u, w, w, t), M^2(w, w, w, t) \\
M^2(w, u, u, t), M^2(u, w, w, t)
\end{array} \right) > 1
$$

$M^2(u, w, w, kt) \geq M^2(u, w, w, t)$. Therefore $M(u, w, w, kt) \geq M(u, w, w, t)$, we get $u = w$. That is
w is a unique common fixed point of A, B, G, H, S, T, P, Q and R in X. $\square$

Remark 3.1. If we take $B = T = H = I$ identity map on X in above theorem then we get the following corollary

Corollary 3.2. *Let $(X, M, *)$ be a complete generalized fuzzy metric space and A, G, S, P, Q and R be mappings from X into itself such that the following conditions are satisfied:*

(3.2.1) $P(X) \subset S(X), Q(X) \subset A(X)$ and $R(X) \subset G(X)$

(3.2.2) (P, A) is compatible of type (β) and $(Q, G), (R, S)$ are weak compatible.

(3.2.3) There exists $k \in (0, 1)$ such that for every $x, y, z \in X$ and $t > 0$

$$
F \left(\begin{array}{c}
M^2(Px, Qy, Rz, kt), M^2(Ax, Sz, Gy, t) \\
M^2(Px, Sz, Gy, t), M^2(Rz, Qy, Sz, t) \\
M^2(Gy, Px, Ax, t), M^2(AxQy, Rz, t)
\end{array} \right) \geq 1.
$$

Then A, G, T, P, Q and R have a unique common fixed point in X.

Remark 3.2. If we take weakly compatible mapping in place of compatible mapping of type (β) then we get following result.

Corollary 3.3. *Let $(X, M, *)$ be a complete generalized fuzzy metric space and A, B, G, H, S, T, P, Q and R be mappings from X into itself such that the following conditions are satisfied:*

(3.3.1) $P(X) \subset ST(X), Q(X) \subset AB(X)$ *and* $R(X) \subset GH(X)$,

(3.3.2) $(P, AB), (Q, GH)$ *and* (R, ST) *are weak compatible.*

(3.3.3) *There exists $k \in (0, 1)$ such that for every $x, y, z \in X$ and $t > 0$*

$$
F \left(\begin{array}{c}
M^2(Px, Qy, Rz, kt), M^2(ABx, STz, GHy, t) \\
M^2(Px, STz, GHy, t), M^2(Rz, Qy, STz, t) \\
M^2(GHy, Px, ABx, t), M^2(ABxQy, Rz, t)
\end{array} \right) \geq 1.
$$

Then A, B, G, H, S, T, P, Q and R have a unique common fixed point in X.

4 Conclusions

In this article, We have obtained new common fixed point theorems for generalized fuzzy metric spaces with the help of compatible maps of type (β) and weakly compatible by using an implicit relation. Further, fixed point results can be extended with various abstract spaces such as partial fuzzy metric spaces, bipolar fuzzy metric space and neutrosophic metric spaces by using different types of compatible maps.

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Competing Interests

Authors have declared that no competing interests exist.

References

[1] Zadeh LA. Fuzzy sets. Inform. and Control. 1965;8:338-353.

[2] Kramosil I, Michalek J. Fuzzy metric and statistical spaces. Ky-Bernetica. 1975;11:336-344.

[3] George A, Veeramani P. On some results in fuzzy metric spaces. Fuzzy Sets and Systems. 1994;64:395-399.

[4] Sedghi S, Shobe N. Fixed point theorem in M -fuzzy metric spaces with property (E). Advances in Fuzzy Mathematics. 2006;1(1):55-65.

[5] Veerapandi T, Jeyaraman M, Paul Raj Joseph J. Some fixed point and coincident point theorem in generalized M-fuzzy metric space. Int. Journal of Math. Analysis. 2009;3:627-635.

[6] Mishra SN, Sharma N, Singh SL. Common fixed points of maps on fuzzy metric spaces. Internet. J. Math, Math. Sci. 1994;17:253-258.

[7] Pant V, Pant RP. Fixed points in fuzzy metric space for noncompatible maps. Soochow J. Math. 2007;33(4):647-655.

[8] Cho J, Pathak HK, Kang SM, Jung JS. Common fixed points of compatible maps of type (β) on fuzzy metric spaces. Fuzzy Sets and Systems. 1998;93:99-111.

[9] Altun I, Turkoglu D. Some fixed point theorems on fuzzy metric spaces with implicit relations, commun. Korean Math. Soc. 2008;23(1):111-124.

[10] Schweizer B, Sklar A. Statistical metric spaces. Pacific J. Math. 1960;10:313-334.

[11] Singh B, Jain S. Weak compatibility and fixed points theorems in Fuzzy metric spaces. Ganita. 2005;56(2):167-176.