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On the Characteristics and Application of Inverse Power Pranav Distribution

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

In this article, we study the mathematical characteristics of the inverse power Pranav distribution. The proposed distribution has three special cases namely Pranav, inverse Pranav and inverse power Pranav distributions. In addition with the basic properties of the distribution, the maximum likelihood method was employed in computing the parameters of the distribution. The 95% confidence interval was estimated for each of the parameters and finally, the distribution was applied to 128 bladder cancer patients to illustrate its applicability, and compared to Pranav distribution, inverse power Lindley distribution and inverse Ishita distribution. However, the inverse power Pranav distribution proved superiority over the competing models.

Keywords: *Pranav distribution; inverse power pranav distribution; stochastic ordering; exponentiated inverse power pranav distribution; goodness of fit test.*

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1 Introduction

In statistical theory, the modelling and analysis of life data is a significant facet of statistical work. In many applied sciences, statistical distributions are needed for modelling lifetime datasets with monotone and non-monotone hazard rates. Oftentimes, such datasets are seen in biological sciences, engineering, insurance and finance, amid others. Modelling of these datasets, aid the description of the event from which the datasets were obtained. The shape of the data gives explicit description and understanding of the event under study. Thus, appropriate decision can be taken base on the results from the analysis. From a different look, the failure behaviour of any system can be deemed as a random variable owing to the distinctions from one system to another consequential from the nature of the system. Consequently, it appears logical to find a statistical model that can fit appropriately, the failure of the system. Also, survival data are classified by their hazard rate, for instance, the number of deaths per unit in a period of time, which can be monotone or non-monotone. For modelling such data, many lifetime distributions have been developed based on hazard rate. One of such distribution is the Pranav distribution proposed by [1] with probability density function (pdf) and cumulative density function (cdf) respectively given by

$$f(x,\alpha)=\frac{\alpha^4}{\alpha^4+6}\left(\alpha+x^3\right)e^{-\alpha x};x>0,\alpha>0 \tag{1.1}$$

$$F(x,\alpha)=1-\left[1+\frac{\alpha x\left(\alpha^2x^2+3\alpha x+6\right)}{\alpha^4+6}\right]e^{-\alpha x},x>0,\alpha>0 \tag{1.2}$$

The mathematical and statistical characteristics of this distribution and method of estimation have been explicitly derived in the paper. The distribution was subjected to two life datasets from engineering and found to be superior over Lindley, exponential, Akash, Ishita, Shanker and Sujatha distributions. [2] developed the inverse Lindley distribution and provided the properties of the proposed distribution including the estimation method. They further proposed its applicability as a stress-strength reliability model for survival data analysis. The estimation of stress-strength parameters and, the stress-strength reliability were approached by both classical and Bayesian paradigms. Two real data sets representing survival of Head and Neck cancer patients were fitted using the inverse Lindley distribution and compared with inverse Rayleigh distribution. Inverse Rayleigh distribution was found to perform less than the inverse Lindley distribution.

In another study, [3] proposed a new three-parameter inverse distribution, called extended inverse Lindley distribution. He argued that the model has more flexibility than other types of inverse distributions due to the shape of its density as well as its hazard rate functions. They derived the pdf, cdf, hazard rate function, the moments, moment generating function, and the quantile function in simple mathematical forms. Maximum likelihood estimation of the parameters and their asymptotic standard distribution and confidence interval were estimated. The measure of the uncertainty such as Rényi was also derived. Application of the model to a real data set is presented and compared to other extension of inverse Lindley and inverse Weibull distributions, such as inverse Lindley, generalized inverse Lindley, inverse Weibull and generalized inverse Weibull. Other related literatures on this include the articles by [4-6], among others. In this study, we aim at providing the mathematical characteristics of the distribution proposed by [7] and to demonstrate its applicability using a lifetime data set. The rest of the paper is organized as follows. In Section 2, the inverse power Pranav (IPP) distribution and the graphs exhibiting the behaviour of distribution for varying values of the parameters. In Section 3, the survival and hazard functions were presented including their curves. Section 4 contains the mathematical characteristics of the inverse power Pranav distribution, the numerical application of the distribution to bladder cancer and estimation of the 95% confidence intervals for the parameters. Finally, in Section 5 the work is crowned with conclusion.

2 Inverse Power Pranav Distribution

[8] proposed a new distribution known as the inverse power Pranav distribution. The density function of the inverse power Pranav distribution is expressed as

$$f_{IPP}\left(x,\alpha,\beta\right)=\frac{\beta\alpha^{4}}{\alpha^{4}+6}\left(\alpha+x^{-3\beta}\right)x^{-(\beta+1)}e^{-\alpha x^{-\beta}};x>0,\alpha,\beta>0 \tag{2.1}$$

Remark: for $\beta=1$, the inverse power Pranav distribution returns to a one parameter inverse Pranav distribution defined by the pdf

$$f_{IP}\left(x,\alpha\right)=\frac{\alpha^{4}}{\alpha^{4}+6}\left(\alpha+x^{-3}\right)x^{-2}e^{-\alpha x^{-1}};x>0,\alpha>0 \tag{2.2}$$

The cdf of the inverse power Pranav distribution in (2.1) is given by

$$F_{IP}\left(x,\alpha,\beta\right)=\left\{1+\frac{\alpha x^{-\beta}\left(\alpha^{2}x^{-2\beta}+3\alpha x^{-\beta}+6\right)}{\alpha^{4}+6}\right\}e^{-\alpha x^{-\beta}};x>0,\alpha,\beta>0 \tag{2.3}$$

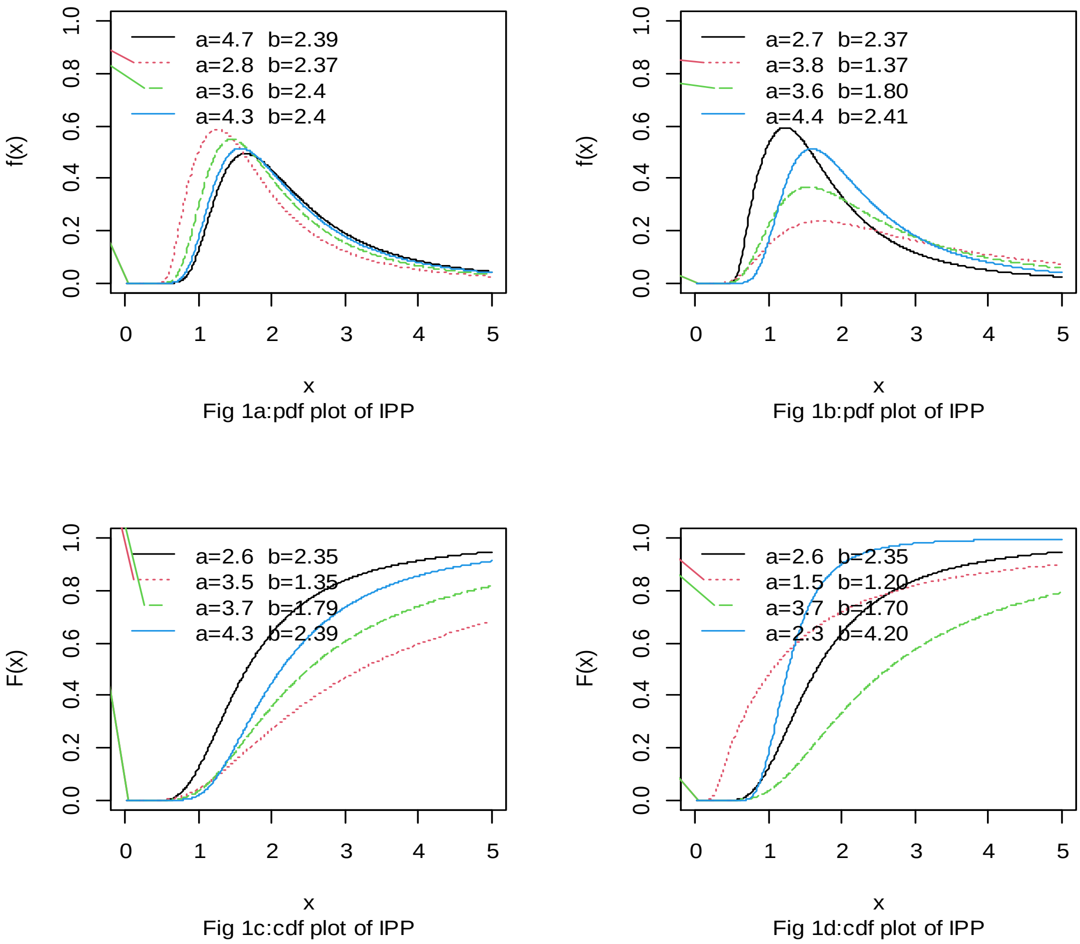

Fig. 1a, 1b, 1c and 1d show the behaviour of pdf and cdf of the proposed distribution; A close look at the plots reveals that the distribution has an increasing and decreasing function

3 Survival and Hazard Function

For a distribution with cdf $F\left(x;\alpha\right)$, the survival function $S\left(x;\alpha\right)$ is defined as

$$S(x;\alpha) = 1 - F(x;\alpha) \tag{3.1}$$

Consequently, the survival function of the inverse power Pranav distribution is

$$S(x;\alpha) = 1 - \left\{1 + \frac{\alpha x^{-\beta}\left(\alpha^2 x^{-2\beta} + 3\alpha x^{-\beta} + 6\right)}{\alpha^4 + 6}\right\} e^{-\alpha x^{-\beta}} \tag{3.2}$$

And the hazard function $h(x;\alpha)$ of inverse power Pranav distribution is giving by

$$h(x;\alpha) = \frac{f(x;\alpha)}{1 - F(x;\alpha)} \tag{3.3}$$

$$h(x;\alpha) = \frac{\frac{\beta\alpha^4}{\alpha^4 + 6}\left(\alpha + x^{-3\beta}\right) x^{-(\beta+1)} e^{-\alpha x^{-\beta}}}{1 - \left\{1 + \frac{\alpha x^{-\beta}\left(\alpha^2 x^{-2\beta} + 3\alpha x^{-\beta} + 6\right)}{\alpha^4 + 6}\right\} e^{-\alpha x^{-\beta}}} \tag{3.4}$$

The nature of survival and hazards functions are shown in Fig. 2a, 2b, 2c and 2d, for varying values of α and β

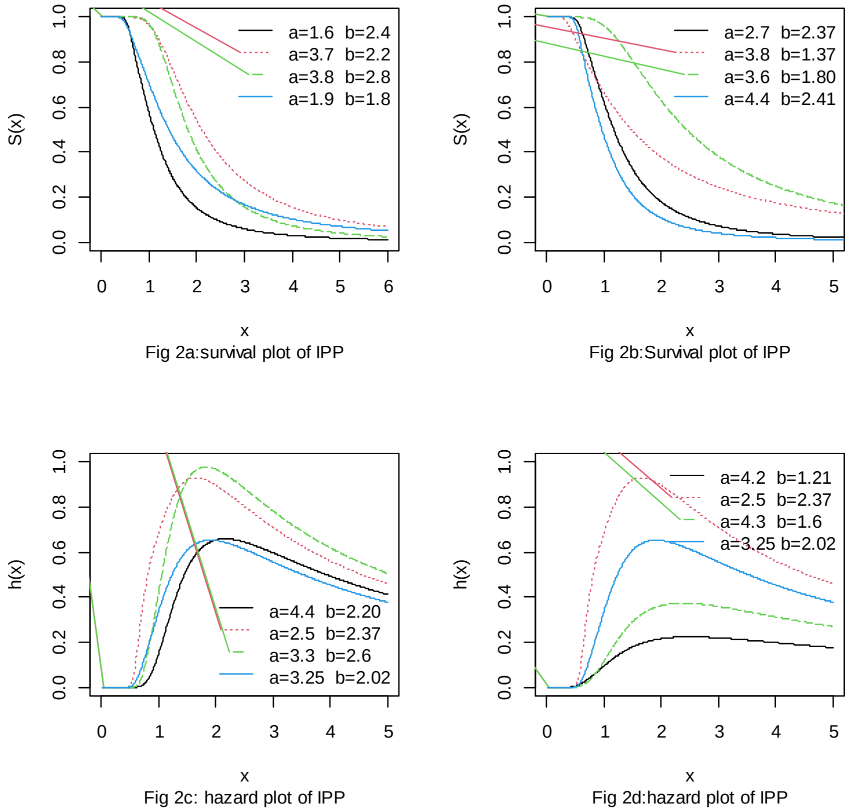

Fig. 2. Graph of survival and hazard function of IPP

4 Properties of Inverse Power Pranav Distribution

4.1 Moments

In order to study the significant characteristics of any distribution, the moment of the distribution under study must be found. Moments enable one study important features of a distribution such as mean, variance, skewness and kurtosis. The most important aspect of a moment is the *rth* moment. This is because; it enables easy derivation of other moments.

Theorem 4.1: Let X be a random variable that follows inverse power Pranav distribution with parameters (α,β). Then, the *rth* moment is given as

$$E\left(X^r\right)=\mu_r^{'}=\frac{\alpha^{r/\beta}}{\left(\alpha^4+6\right)}\left(\alpha^4\Gamma\left(1-r/\beta\right)+\Gamma\left(4-r/\beta\right)\right)\quad r\le 3 \tag{4.1}$$

The *rth* moment of a distribution is given by the expression

$$E\left(X^r\right)=\int_0^\infty x^r f\left(x\right)dx \tag{4.2}$$

$$=\int_0^\infty x^r\frac{\beta\alpha^4}{\alpha^4+6}\left(\alpha+x^{-3\beta}\right)x^{-(\beta+1)}e^{-\alpha x^{-\beta}}dx$$

$$=\frac{\beta\alpha^4}{\alpha^4+6}\left\{\alpha\int_0^\infty\frac{e^{-\alpha x^{-\beta}}}{x^{\beta-r+1}}dx+\int_0^\infty\frac{e^{-\alpha x^{-\beta}}}{x^{4\beta-r+1}}dx\right\} \tag{4.3}$$

By letting $t=x^\beta$, applying the transformation technique gives and substituting appropriately in eq.(4.3) gives

$$E\left(X^r\right)=\frac{\beta\alpha^4}{\alpha^4+6}\left\{\alpha\int_0^\infty\frac{e^{-\alpha/t}}{t^{2-r/\beta}}dt+\int_0^\infty\frac{e^{-\alpha/t}}{t^{4-r/\beta}}dt\right\} \tag{4.4}$$

Recall that $\int_0^\infty\frac{e^{-\beta/t}}{x^{\alpha+1}}dx=\frac{\Gamma(\alpha)}{\beta^\alpha}$, applying this to eq. (4.4), we get

$$E\left(X^r\right)=\frac{\beta\alpha^4}{\alpha^4+6}\left\{\alpha\frac{\Gamma\left(1-r/\beta\right)}{\alpha^{\left(1-r/\beta\right)}}+\frac{\Gamma\left(4-r/\beta\right)}{\alpha^{\left(4-r/\beta\right)}}\right\} \tag{4.5}$$

Simplifying further, one obtains the *rth* moment of the inverse Pranav distribution as

$$E\left(X^r\right)=\mu_r^{'}=\frac{\alpha^{r/\beta}}{\left(\alpha^4+6\right)}\left(\alpha^4\Gamma\left(1-r/\beta\right)+\Gamma\left(4-r/\beta\right)\right);r\le 3 \tag{4.6}$$

By substituting for $r=1\text{and}\,2$, we get the first and second crude moments $\mu_1^{'}$ and $\mu_2^{'}$. $\mu_1^{'}$ is the mean given by

$$\mu_1^{'} = \frac{\alpha^{1/\beta}}{\left(\alpha^4 + 6\right)}\left(\alpha^4 \Gamma\left(1 - 1/\beta\right) + \Gamma\left(4 - 1/\beta\right)\right)$$

and

$$\mu_2^{'} = \frac{\alpha^{2/\beta}}{\left(\alpha^4 + 6\right)}\left(\alpha^4 \Gamma\left(1 - 2/\beta\right) + \Gamma\left(4 - 2/\beta\right)\right)$$

The variance of the inverse power Pranav distribution can be obtained as follows

$$\mu_2 = \mu_2^{'} - \left(\mu_1^{'}\right)^2 = \frac{\alpha^{2/\beta}}{\left(\alpha^4 + 6\right)}\left(\alpha^4 \Gamma\left(1 - 2/\beta\right) + \Gamma\left(4 - 2/\beta\right)\right) - \left(\frac{\alpha^{1/\beta}}{\left(\alpha^4 + 6\right)}\left(\alpha^4 \Gamma\left(1 - 1/\beta\right) + \Gamma\left(4 - 1/\beta\right)\right)\right)^2 \quad (4.7)$$

4.2 Moment Generating Function

Theorem 4.2: Given a random variable X, such that $X \sim IPP(\alpha, \beta)$, the moment generating function is given by

$$M_x(t) = \frac{\alpha^{r/\beta}}{k!\left(\alpha^4 + 6\right)} \sum_{k=0}^{\infty} t^k \left(\alpha^4 \Gamma\left(1 - r/\beta\right) + \Gamma\left(4 - r/\beta\right)\right) \quad (4.8)$$

For a continuous distribution, the moment generating function is

$$M_x(t) = E\left(e^{tX}\right) = \int_0^{\infty} e^{tx} f_{IPP}\left(x; \alpha, \beta\right) dx \quad (4.9)$$

Taylor's series expansion of e^{tx} gives

$$M_x(t) = \int_0^{\infty} \left(1 + tx + \frac{(tx)^2}{2!} + \cdots\right) f_{IPP}\left(x; \alpha, \beta\right) dx$$

$$= \int_0^{\infty} \sum_{k=0}^{\infty} \frac{t^k}{k!} x^k f\left(x; \alpha, \beta\right) dx$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{k!} \int_0^{\infty} x^k f\left(x; \alpha, \beta\right) dx = \sum_{k=0}^{\infty} \frac{t^k}{k!} E\left(X^k\right)$$

Where $E\left(X^k\right) = \mu_r^{'}$. Thus, the moment generating function of IPP distribution becomes

$$M_x(t) = \frac{\alpha^{r/\beta}}{k!\left(\alpha^4 + 6\right)} \sum_{k=0}^{\infty} t^k \left(\alpha^4 \Gamma\left(1 - r/\beta\right) + \Gamma\left(4 - r/\beta\right)\right) \quad (4.10)$$

4.3 Distribution of Order Statistics and Quantile Function of Inverse Power Pranav Distribution

Theorem 4.3: Suppose X is a random variable that follows inverse Power Pranav distribution with parameters (α,β), having the pdf and cdf giving in (2.1) and (2.3), then the pdf and cdf of the ω order statistics are giving by

$$f_{X_\omega}(x)=\frac{\beta\alpha^4 n!\left(\alpha+x^{-3\beta}\right)}{(\omega-1)!(n-\omega)!\left(\alpha^4+6\right)^{j+1}}e^{-\alpha(1+j)x^{-\beta}}\sum_{i=0}^{n-\omega}\binom{n-\omega}{i}(-1)^i\sum_{j=0}^{\infty}\binom{\omega+i-1}{j}\sum_{k=0}^{\infty}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}3^k 2^l\alpha^{3j-k-l}x^{2\beta k-3\beta j-\beta-1}$$

And

$$F_{X_\omega}(x)=\sum_{i=\omega}^{n}\binom{n}{i}\sum_{j=0}^{n-i}\binom{n-i}{j}(-1)^j\sum_{k=0}^{\infty}\binom{i+j}{k}\left(\frac{\alpha x^{-\beta}}{\alpha^4+6}\right)^k\left(\alpha^2x^{-2\beta}+3\alpha x^{-\beta}+6\right)^k e^{-k\alpha x^{-\beta}}$$

To provide the proof for $f_{X_\omega}(x)$, we recall that

$$f_{X_\omega}(x)=\frac{n!}{(\omega-1)!(n-\omega)!}F^{\omega-1}(x)\left(1-F(x)\right)^{n-\omega}f(x) \tag{4.11}$$

Inserting eq. (2.1) and (2.3) into (4.11), with little binomial expansion, we get

$$\begin{aligned}f_{X_\omega}(x)&=\frac{\beta\alpha^4 n!\left(\alpha+x^{-3\beta}\right)x^{-(\beta+1)}e^{-\alpha x^{-\beta}}}{(\omega-1)!(n-\omega)!\left(\alpha^4+6\right)}\sum_{i=0}^{n-\omega}\binom{n-\omega}{i}(-1)^i\\&\times\left\{\left(1+\frac{\alpha x^{-\beta}\left(\alpha^2x^{-2\beta}+3\alpha x^{-\beta}+6\right)}{\alpha^4+6}\right)e^{-\alpha x^{-\beta}}\right\}^{\omega+i-1}\end{aligned} \tag{4.12}$$

The series expansion of

$$\begin{aligned}&\left\{\left(1+\frac{\alpha x^{-\beta}\left(\alpha^2x^{-2\beta}+3\alpha x^{-\beta}+6\right)}{\alpha^4+6}\right)e^{-\alpha x^{-\beta}}\right\}^{\omega+i-1}\\&=\sum_{j=0}^{\infty}\binom{\omega+i-1}{j}\sum_{k=0}^{\infty}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\frac{\alpha^{3j-k-l}3^k2^l}{\left(\alpha^4+6\right)^j}x^{2\beta k-3\beta j}e^{-\alpha jx^{-\beta}}\end{aligned} \tag{4.13}$$

Substituting (4.13) into (4.12), we obtain the pdf of the ωth order statistics of inverse power Pranav distribution. Thus,

$$\begin{aligned}f_{X_\omega}(x)&=\frac{\beta\alpha^4 n!\left(\alpha+x^{-3\beta}\right)}{(\omega-1)!(n-\omega)!\left(\alpha^4+6\right)^{j+1}}e^{-\alpha(1+j)x^{-\beta}}\sum_{i=0}^{n-\omega}\binom{n-\omega}{i}(-1)^i\sum_{j=0}^{\infty}\binom{\omega+i-1}{j}\\&\times\sum_{k=0}^{\infty}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}3^k2^l\alpha^{3j-k-l}x^{2\beta k-3\beta j-\beta-1}\end{aligned} \tag{4.14}$$

Also, for the cdf of the order statistics, as defined in [9] is given by

$$F_{X_\omega}(x) = \sum_{i=\omega}^{n} \binom{n}{i} F^i(x)\left(1-F(x)\right)^{n-i} \tag{4.15}$$

$$= \sum_{i=\omega}^{n} \binom{n}{i} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j \left\{ \left(1 + \frac{\alpha x^{-\beta}\left(\alpha^2 x^{-2\beta} + 3\alpha x^{-\beta} + 6\right)}{\alpha^4 + 6} \right) e^{-\alpha x^{-\beta}} \right\}^{i+j} \tag{4.16}$$

Following similar method, equation (4.16) becomes

$$F_{X_\omega}(x) = \sum_{i=\omega}^{n} \binom{n}{i} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j \sum_{k=0}^{\infty} \binom{i+j}{k} \left(\frac{\alpha x^{-\beta}}{\alpha^4 + 6} \right)^k \left(\alpha^2 x^{-2\beta} + 3\alpha x^{-\beta} + 6\right)^k e^{-k\alpha x^{-\beta}} \tag{4.17}$$

4.3.1 Quantile function

In order to generate random numbers, we use the quantile function. Given the cdf, $F(x)$, the qth quantile p is the value of the random variable X defined by

$$p = F^{-1}\left(x_p\right) \text{for } 0 < p < 1 \tag{4.18}$$

Plugging in $F_{IPP}(x)$ in (4.18), we get

$$p = \left\{ \left(1 + \frac{\alpha x^{-\beta}\left(\alpha^2 x^{-2\beta} + 3\alpha x^{-\beta} + 6\right)}{\alpha^4 + 6} \right) e^{-\alpha x^{-\beta}} \right\}^{-1} \tag{4.19}$$

Simplifying equation (4.19), we obtain the quantile function of inverse power Pranav distribution. Consequently, we have

$$x_p = \left\{ \frac{1}{\alpha} \ln \left(p + \frac{p\alpha x_p^{-\beta}\left(\alpha^2 x_p^{-2\beta} + 6\right)}{\left(\alpha^4 + 6\right)} \right) \right\}^{-1/\beta} \tag{4.20}$$

4.4 Stochastic Ordering

In order to compare the behaviour of positive continuous random variables, we use stochastic ordering. As stated by [10], a random variable X is supposed to be lesser than a random variable Y in the

a) Stochastic order $\left(X \le_{st} Y\right)$ if $F_X(x) \ge F_Y(x); \forall x$
b) Hazard rate order $\left(X \le_{hr} Y\right)$ if $h_X(x) \ge h_Y(x); \forall x$
c) Mean residual life order $\left(X \le_{mrl} Y\right)$ if $m_X(x) \ge m_Y(x); \forall x$
d) Likelihood ratio order $\left(X \le_{lr} Y\right)$ if $\frac{f_X(x;\alpha,\theta)}{f_Y(x;\alpha,\theta)}$ decreases in x

The results established by [11] is as follows

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{mrl} Y \Rightarrow X \leq_{st} Y$$

The IPP distribution is well-ordered based on the one with the robust likelihood ratio.

Theorem 4.4. Suppose $X \sim IPP(\alpha_1, \beta_1)$ and $Y \sim IPP(\alpha_2, \beta_2)$. If $\alpha_1 > \alpha_2$ and $\beta_2 \geq \beta_1$, then $X \leq_{lr} Y$. Hence, $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$.

To establish the proof, the likelihood ratio is

$$\frac{f_X(x)}{f_Y(x)} = \frac{\beta_1 \alpha^4 \Big/ \left(\alpha_1^4 + 6\right)\left(\alpha_1 + x^{-3\beta_1}\right) x^{-(\beta_1+1)} e^{-\alpha_1 x^{-\beta_1}}}{\beta_2 \alpha^4 \Big/ \left(\alpha_2^4 + 6\right)\left(\alpha_2 + x^{-3\beta_2}\right) x^{-(\beta_2+1)} e^{-\alpha_2 x^{-\beta_2}}} \tag{4.21}$$

If $\beta_1 = \beta_2 = \beta$, we have

$$\frac{f_X(x)}{f_Y(x)} = \left\{\frac{\alpha_1^4\left(\alpha_2^4 + 6\right)}{\alpha_2^4\left(\alpha_1^4 + 6\right)}\right\}\left(\frac{\alpha_1 + x^{-3\beta}}{\alpha_2 + x^{-3\beta}}\right) e^{-(\alpha_1 - \alpha_2)x^{-\beta}}$$

Taking the natural logarithm of both sides, we obtain

$$\ln \frac{f_X(x)}{f_Y(x)} = \ln\left\{\frac{\alpha_1^4\left(\alpha_2^4 + 6\right)}{\alpha_2^4\left(\alpha_1^4 + 6\right)}\right\} + \ln\left(\frac{\alpha_1 + x^{-3\beta}}{\alpha_2 + x^{-3\beta}}\right) - \left(\alpha_1 - \alpha_2\right) x^{-\beta}$$

$$\frac{d}{dx} \ln \frac{f_X(x)}{f_Y(x)} = \frac{-3\beta x^{-3\beta-1}}{\alpha_1 + x^{-3\beta}} + \frac{3\beta x^{-3\beta-1}}{\alpha_2 + x^{-3\beta}} + \beta\left(\alpha_1 - \alpha_2\right) x^{-\beta-1}$$

Further simplification, one arrive at the following results

$$\frac{d}{dx} \ln \frac{f_X(x)}{f_Y(x)} = \frac{3\beta\left(\alpha_1 - \alpha_2\right) x^{-(3\beta+1)}}{\left(\alpha_1 + x^{-3\beta}\right)\left(\alpha_2 + x^{-3\beta}\right)} + \beta\left(\alpha_1 - \alpha_2\right) x^{-(\beta+1)} \tag{4.22}$$

For $\beta_1 = \beta_2$ and $\alpha_1 > \alpha_2$, $\frac{d}{dx} \ln \frac{f_X(x)}{f_Y(x)} < 0$. This justify the proof that $X \leq_{lr} Y$ and $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$ **.**

4.5 Entropy of IPP Distribution

The entropy of a random variable X measures the degree of uncertainty of distribution. The larger the entropy value, the greater the uncertainty in the data. There are several kinds of entropy, here only two will be considered. The Rényi entropy and Tsallis Entropy.

4.5.1 Rényi and Tsallis entropy of IPP distribution

Theorem 4.5: Suppose $X \sim IPP(\alpha, \beta)$, the Rényi and Tsallis entropy are defined by

$$\ell(\eta)=\frac{1}{1-\eta}\log\left\{\left(\frac{\beta\alpha^4}{\alpha^4+6}\right)^{\eta}\sum_{i=0}^{\infty}\binom{n}{i}\alpha^{\eta-i}\beta^{-1}\frac{\Gamma\left(\eta+\eta/\beta+3i-1/\beta\right)}{(\alpha\eta)^{\left(\eta+\eta/\beta+3i-1/\beta\right)}}\right\}$$

$$S_\lambda=\frac{1}{\lambda-1}\left\{1-\left(\frac{\beta\alpha^5}{\alpha^4+6}\right)^{\lambda}\sum_{j=0}^{\infty}\binom{\lambda}{j}\alpha^{-j}\beta^{-1}\frac{\Gamma\left(3j+\lambda+\lambda/\beta-1/\beta+1\right)}{(\alpha\lambda)^{\left(3j+\lambda+\lambda/\beta-1/\beta+1\right)}}\right\}$$

Rényi entropy is defined as

$$\ell(\eta)=\frac{1}{1-\eta}\log\left(\int_R f_R^{\eta}(x)dx\right)\quad \text{where } \eta>0 \text{ and } \eta\neq 1 \text{ [12]} \tag{4.23}$$

$$=\frac{1}{1-\eta}\log\int_0^{\infty}\left(\frac{\beta\alpha^4}{\alpha^4+6}\left(\alpha+x^{-3\beta}\right)x^{-(\beta+1)}e^{-\alpha x^{-\beta}}\right)^{\eta}dx$$
(4.24)

$$=\frac{1}{1-\eta}\log\left\{\left(\frac{\beta\alpha^4}{\alpha^4+6}\right)^{\eta}\int_0^{\infty}x^{-\eta(\beta+1)}\alpha^{\eta}\left(1+\alpha^{-1}x^{-3\beta}\right)^{\eta}e^{-\alpha\eta x^{-\beta}}\right\}dx \tag{4.25}$$

Applying binomial series to eq. (4.25), we get

$$\ell(\eta)=\frac{1}{1-\eta}\log\left\{\left(\frac{\beta\alpha^4}{\alpha^4+6}\right)^{\eta}\int_0^{\infty}x^{-\eta(\beta+1)}\alpha^{\eta}\sum_{i=0}^{\infty}\binom{\eta}{i}\alpha^{-i}x^{-3\beta i}e^{-\alpha\eta x^{-\beta}}\right\} \tag{4.26}$$

If we let $t=x^{-\beta}$, a careful simplification will give

$$\ell(\eta)=\frac{1}{1-\eta}\log\left\{\left(\frac{\beta\alpha^4}{\alpha^4+6}\right)^{\eta}\sum_{i=0}^{\infty}\binom{\eta}{i}\alpha^{\eta-i}\beta^{-1}\int_0^{\infty}t^{-\left(\eta+\eta/\beta+3i-1/\beta+1\right)}e^{-\alpha\eta/t}dt\right\} \tag{4.27}$$

Equation (4.27) can be transformed to gamma function. Thus, we have

$$\ell(\eta)=\frac{1}{1-\eta}\log\left\{\left(\frac{\beta\alpha^4}{\alpha^4+6}\right)^{\eta}\sum_{i=0}^{\infty}\binom{\eta}{i}\alpha^{\eta-i}\beta^{-1}\left(\frac{\Gamma\left(\eta+\eta/\beta+3i-1/\beta\right)}{(\alpha\eta)^{\left(\eta+\eta/\beta+3i-1/\beta\right)}}\right)\right\} \tag{4.28}$$

Similarly, Tsallis entropy is obtained as follows

$$S_\lambda=\frac{1}{\lambda-1}\left(1-\int_0^{\infty}f_R^{\lambda}(x)dx\right)\text{ [13]} \tag{4.29}$$

$$S_\lambda=\frac{1}{\lambda-1}\left(1-\int_0^{\infty}\left(\frac{\beta\alpha^4}{\alpha^4+6}\left(\alpha+x^{-3\beta}\right)x^{-(\beta+1)}e^{-\alpha x^{-\beta}}\right)^{\lambda}dx\right)$$

$$= \frac{1}{\lambda-1}\left(1-\left(\frac{\beta\alpha^4}{\alpha^4+6}\right)^{\lambda}\alpha^{\lambda}\int_0^{\infty}\left(\left(1+\frac{x^{-3\beta}}{\alpha}\right)^{\lambda}x^{-\lambda(\beta+1)}e^{-\lambda\alpha x^{-\beta}}\right)dx\right) \quad (4.30)$$

By applying binomial expansion theorem in equation (4.30), and simplifying, we obtain the following

$$S_{\lambda} = \frac{1}{\lambda-1}\left(1-\left(\frac{\beta\alpha^5}{\alpha^4+6}\right)^{\lambda}\sum_{j=0}^{\infty}\binom{\lambda}{j}\alpha^{-j}\int_0^{\infty}x^{-(3\beta j+\lambda\beta+\lambda)}e^{-\lambda\alpha x^{-\beta}}dx\right) \quad (4.31)$$

Applying gamma function, consequently yields

$$S_{\lambda} = \frac{1}{\lambda-1}\left\{1-\left(\frac{\beta\alpha^5}{\alpha^4+6}\right)^{\lambda}\sum_{j=0}^{\infty}\binom{\lambda}{j}\alpha^{-j}\beta^{-1}\frac{\Gamma\left(3j+\lambda+\lambda/\beta-1/\beta+1\right)}{(\alpha\lambda)^{\left(3j+\lambda+\lambda/\beta-1/\beta+1\right)}}\right\} \quad (4.32)$$

4.6 Maximum Likelihood Estimation of the Parameters of IPP Distribution

Suppose $X_1, X_2, X_3, X_4, ..., X_n$ are independent identically distributed random samples of size n with pdf, $f(x_i, \alpha)$, where α is a $(k\times 1)$ vector of parameters that describes $f(x_i, \alpha)$. Then, the combined pdf is

$$f(x_1, x_2, x_3, ..., x_n; \alpha) = \prod_{i=1}^{n} f(x_i; \alpha) \quad (4.33)$$

$$= \prod_{i=1}^{n}\frac{\beta\alpha^4}{\alpha^4+6}\left(\alpha+x^{-3\beta}\right)x^{-(\beta+1)}e^{-\alpha x^{-\beta}} \quad (4.34)$$

The likelihood function denoted by ℓ is given by

$$\ell\left(\alpha \middle| x_1, x_2, x_3, ..., x_n\right) = \left(\frac{\beta\alpha^4}{\alpha^4+6}\right)^n\sum_{i=1}^{n}\left(\alpha+x^{-3\beta}\right)\sum_{i=1}^{n}x^{-(\beta+1)}e^{-\alpha\sum_{i=0}^{n}x^{-\beta}} \quad (4.35)$$

The log-likelihood function of the inverse power Pranav distribution is thus given by

$$L = \ln\ell(x_i, \alpha, \beta) = n\ln\beta + 4n\ln\alpha - n\ln\left(\alpha^4+6\right) + \sum_{i=1}^{n}\ln\left(\alpha+x^{-3\beta}\right) - (\beta+1)\sum_{i=1}^{n}\ln x - \alpha\sum_{i=0}^{n}x^{-\beta}$$
(4.36)

The maximum likelihood of the estimators is obtained by taking the partial derivative of L with respect to each of the parameters. Consequently, we have

$$\frac{\partial L}{\partial\alpha} = \frac{4n}{\alpha} - \frac{4n\alpha^3}{\alpha^4+6} + \sum_{i=1}^{n}\left(\frac{1+x^{-3\beta}}{\alpha+x^{-3\beta}}\right) - \sum_{i=0}^{n}x^{-\beta} \quad (4.37)$$

$$\frac{\partial L}{\partial\beta} = \frac{n}{\beta} - 3\sum_{i=1}^{n}\frac{x^{-3\beta}\ln x}{\alpha+x^{-3\beta}} - \sum_{i=1}^{n}\ln x + \alpha\sum_{i=1}^{n}x^{-\beta}\ln x \quad (4.38)$$

At $\frac{\partial L}{\partial \alpha} = 0$ and $\frac{\partial L}{\partial \beta} = 0$, we obtain a system of nonlinear equations. Solving the nonlinear equations manually is very monotonous and unwieldy. Hence, R software and for estimating the required parameters.

4.7 Application

The objective of this section is to encourage the use of the IPP distribution by displaying a successful application to modelling lifetime data set. The data set for this study were reported by [14] and [15]. It represents the remission times (in months) of a random sample of 128 bladder cancer patients. The data set is presented as follows

0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46 , 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69, 78.0, 80.0, 83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 111.0, 115.0, 117.0, 125.0, 126.0, 127.0, 129.0, 129.0,139.0, 154.0

Table 1 below shows the summary of the data.

Table 1. Data summary

Minimum	0.080000
Maximum	154.00000
1. Quartile	3.655000
3. Quartile	17.305000
Mean	23.958667
Median	7.605000
Variance	1400.8847
Stdev	37.428395
Skewness	1.938028
Kurtosis	2.352758

In order to assess the effectiveness of the IPP distribution, the data set given above was fitted and compared with three other distributions: Pranav distribution (PD), Power Inverse Lindley distribution (IPL) by [16] and Inverse Ishita distributions by [17]. To assess the goodness-of-fit of above distributions, we employed the Akaike information criterion (AIC), the Bayesian information criterion (BIC) and the Log-likelihood LL, which are computed using the following formula:

Table 2. Parameters, S.E, LL, AIC, BIC, K-S statistics and its p-value of the fitted distributions

Model	Parameters	S.E	LL	AIC	BIC	KS	p
IPP	**α =3.2111**	**0.1923**	**-606.6425**	**1217.285**	**1223.306**	**0.089**	**0.1753**
	β = 0.7078	**0.0361**					
PD	α = 0.1683	0.0069	-947.9402	1897.88	1900.891	0.51794	6.66E-16
	α= 0.6463	0.0364					
IPL	β = 3.2541	0.2367	-609.9978	1223.996	1230.017	0.09936	0.0966
IID	θ = 3.2483	0.212	-642.1729	1286.346	1289.356	0.99885	6.66E-16

- the Akaike information criterion [18] defined by

$$AIC = -2\ln L + 2k ; \tag{4.39}$$

- the Bayesian information criterion [19] defined by

$$BIC = \ln(n)k - 2\ln(\hat{L}) \tag{4.40}$$

where, k is the number of estimable model parameters. $\hat{L}$ is the maximized likelihood of the vector of parameters θ and n is the sample size.The distribution with least AIC, BIC and log-likelihood LL is considered as best. Table 2 above shows the results obtained using R software packages. According to the small values of both AIC, and BIC, the inverse power Pranav distribution performs better than the other competing models. In addition, looking at the large LL value for the IPP distribution, we deduce that it provides a good fit for the given data and hence, it has proved to be the appropriate model. Table 3 below shows the estimates of the parameters and their confidence intervals

Table 3. MLEs of the parameters IPP distribution and their C.I

Model	parameter	S.E	95% confidence interval	
			Lower Limit	Upper Limit
IPP	α=3.2111	0.1923	2.8342	3.588
	β = 0.7078	0.0361	0.637	0.7786
PD	α=0.1683	0.0069	0.1548	0.1818
IPL	α = 0.6463	0.0364	0.575	0.7176
	β = 3.2541	0.2367	2.7902	3.718
IID	θ = 3.2483	0.212	2.8328	3.6638

5 Conclusion

In this paper, we provide the mathematical characteristics of the inverse power Pranav distribution as well as hazard and Survival functions. We discourse a maximum likelihood estimation of the model's parameters. To illustrate the application of the distribution, the IPP distribution was subjected to a real data set, bladder cancer patients. A test of goodness of fit was carried out using K-S Statistics (Kolmogorov-Smirnov Statistics) to determine its superiority over the other competing models. Also, a 95% confidence interval was provided for the parameter estimates as shown in table 3. Each of the estimates for the parameters lies within the confidence limits.

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Competing Interests

Authors have declared that no competing interests exist.

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