Asian Research Journal of Mathematics

5(4): 1-11, 2017; Article no.ARJOM.34602
ISSN: 2456-477X

Stability Analysis and Response Bounds of Gyroscopic Systems

Ubong D. Akpan[1*] and Moses O. Oyesanya[2]

[1]Department of Mathematics and Statistics, Akwa Ibom State University, Ikot Akpaden, Mkpat Enin, Nigeria.
[2]Department of Mathematics, University of Nigeria, Nsukka, Nigeria.

Authors' contributions

This work was carried out in collaboration between both authors. Author UDA designed the study, performed the analysis, wrote the protocol and wrote the first draft of the manuscript. Author MOO proof read the manuscript and effected the corrections. Both authors read and approved the final manuscript.

Article Information

DOI: 10.9734/ARJOM/2017/34602
Editor(s):
(1) Radoslaw Jedynak, Computer Science and Mathematics, Kazimierz Pulaski University of Technology and Humanities, Poland.
Reviewers:
(1) Abdullah Sonmezoglu, Bozok University, Turkey.
(2) Grienggrai Rajchakit, Maejo University, Thailand.
(3) Georgy Omel'yanov, University of Sonora, Mexico.
(4) Qamar Din, The University of Poonch Rawalakot, Pakistan.
Complete Peer review History: http://www.sciencedomain.org/review-history/20229

Original Research Article

Received: 1st June 2017
Accepted: 29th June 2017
Published: 26th July 2017

Abstract

In this work, we develop a stability theorem for determining the stability or otherwise of a gyroscopic system. A Lyapunov function is obtained by solving the arising Lyapunov matrix equation. The Lyapunov function is then used to obtain response bounds for displacements and velocities both in the homogeneous and inhomogeneous cases. Examples are given to illustrate the efficacy of the results obtained.

Keywords: Stability; gyroscopic system; Lyapunov function; Lyapunov matrix equation, response bounds.

1 Introduction

Gyroscopic effects play an important role in many problem areas of science and engineering. Systems of the form

$$M\ddot{x} + D\dot{x} + Kx = f(t) \tag{1}$$

*Corresponding author: E-mail: akpanubong2210@yahoo.com;

describe gyroscopic systems (where M and K are Hermitian matrices). The mass matrix M and the stiffness matrix K are positive definite($M = M^* > 0, K = K^* > 0$), where * denotes the conjugate transpose. The matrix G of the gyroscopic force is skew-Hermitian ($G = -G^*$) in particular real skew-symmetric. The vector x represents the generalized co-ordinates of the system and $f(t)$ describes the excitation.

The stability or otherwise of matrix second-order systems has been of considerable interest for over three decades. These systems, which are of the form (1) are of fundamental importance in the study of vibrational phenomena. These systems are important mathematical models for rotor systems, satellites and fluid conveying pipes. Stability properties of the systems have been studied for more than one hundred years.

These systems have been studied by [1-10] and useful results for establishing the stability or instability of the systems are given.

In this work, we study the gyroscopic system with a view to obtaining a novel condition for determining the stability or instability of the systems.

2 Preamble

Consider the homogeneous linear system obtained from (1)

$$M\ddot{x} + G\dot{x} + Kx = 0 \tag{2}$$

Assuming solutions of the form $x = qe^{\lambda t}$

(where q is an arbitrary constant and λ is the eigenvalue),

Using $x = qe^{\lambda t}$ on (2) we have

$$\left(\lambda^2 M + \lambda G) + K\right)q = 0 \tag{3}$$

where $e^{\lambda t} \neq 0$ and $q \neq 0$.

The stability of the system (2) can be understood in terms of the eigenvalue problem (3). The eigenvalues λ are obviously the roots of the characteristics polynomial of degree *2n*, det $\left(\lambda^2 M + \lambda G + K\right)$. If all eigenvalues have negative real parts, then the system (2) is said to be asymptotically stable. The asymptotic stability of a system can also be determined by Routh-Hurwitz Criterion.

Alternatively, the stability of the system can be discussed directly by such properties of the system matrices which can be interpreted in a physical way. Applying the direct method of Lyapunov, such an interpretation is usually possible.

The system (2) is equivalent to the system

$$\dot{z} = Az \tag{4}$$

where

$$\dot{z}=\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}=\begin{pmatrix}\dot{x}_1\\ \ddot{x}_1\end{pmatrix},\qquad z=\begin{pmatrix}x_1\\ \dot{x}_1\end{pmatrix}=\begin{pmatrix}x_1\\ x_2\end{pmatrix}\quad A=\begin{bmatrix}O & I\\ -M^{-1}K & -M^{-1}G\end{bmatrix}$$

where I is the identity matrix.

The function *V(z(t))* is called a Lyapunov function for system (4) if $V > 0$ and the time derivative $\dot{V} \le 0$ for all solutions *z(t)* of (4). The existence of such a Lyapunov function implies stability of the system (asymptotic stability if $\dot{V} < 0$) [11]. Lyapunov functions can be considered as generalized energy expressions and therefore it makes sense to look for *V* as a quadratic form in the co-ordinates and in the velocities.

$$V = z(t)^* P z(t) \tag{5}$$

with a Hermitian matrix *P>0*. For the solutions of (4), we then have

$V = z(t)^*(A^*P + PA)\ z(t)$, such that condition $\dot{V} \le 0$ is expressed by the matrix $Q = Q^* \ge 0$ of the Lyapunov matrix equation.

$$A^*P + PA = -Q \tag{6}$$

The system (4) (and therefore also system (2) is asymptotically stable, if there exists Hermitian matrices *P>0* and *Q>0* which satisfy the Lyapunov matrix equation (6). Consider the matrices.

$$P=\begin{bmatrix}K & \frac{\gamma}{2}M\\ \frac{\gamma}{2}M & M\end{bmatrix},\qquad Q=\begin{bmatrix}\gamma K & \frac{\gamma}{2}G\\ \frac{\gamma}{2}G & 2G-\gamma M\end{bmatrix} \tag{7}$$

where γ is a real number.

3 Stability Analysis

The asymptotic stability of the system (4) and of the original system (2) is ensured. Notice that we assume $M = M^* > 0$ and $K = K^* > 0$. We now state the following Lemma which gives a condition for the positive definiteness of P and Q.

Schur's Lemma

A matrix $R=\begin{bmatrix}R_1 & R_2\\ R_2^* & R_3\end{bmatrix}$ with Hermitian submatrices R_1 and R_3 is positive definite if and only if R_1 and $R_3 - R_2^* R_1^{-t} R_2$ are positive definite [12].

Applying the lemma to Q, we get that Q > 0 if and only if there exists $\gamma>0$ such that

$$2G-\gamma M-\frac{\gamma}{2}G*(\gamma K)^{-1}\frac{\gamma}{2}G>0$$

$$\gamma 2G-\gamma^2 M-\frac{\gamma^2}{4}G*K^{-1}G>0$$

Rearranging terms we get the following condition

$$-\gamma^2\left(M+\frac{1}{4}G*K^{-1}G\right)+\gamma 2G>0 \tag{8}$$

Consider all $z\in C^n$, then (8) is equivalent to the inequality

$$-\gamma^2 z*\left(M+\frac{1}{4}G*K^{-1}G\right)z+\gamma z*2Gz>0 \tag{9}$$

Taking $z^*z=1$, the coefficients of the quadratic polynomial in γ are Rayleigh quotients for Hermitian matrices. These Rayleigh quotients are limited by the smallest eigenvalue λ_{min} and the largest eigenvalue λ_{max} of the respective matrices. The Rayleigh quotients for the matrices $M, \frac{1}{4}G*K^{-1}G, 2G$ are all positive since *M*, and K^{-1} are assumed to be positive definite.

Introducing the scalars *a* and *b* defined by

$$\left.\begin{aligned} a&=\lambda_{\max}\left(M+\frac{1}{4}G*K^{-1}G\right)>0\\ b&=\lambda_{\min}2G \end{aligned}\right\} \tag{10}$$

The inequality (9) is now satisfied if there exists $\gamma>0$ with

$$-\gamma^2 a+\gamma b>0 \tag{11}$$

Thus, $\gamma>0$ *or* $\gamma<\frac{b}{a}$

There are solutions of $\gamma>0$ if and only if $b>0$ and $\frac{b^2}{4a}>0$

In this case γ can be chosen as any number in the interval

$$0 < \gamma < \frac{b}{a}$$

The matrices Q and P are positive definite.

which is equivalent to *P>0*

The following theorem is formulated

Theorem 1:

Assume *a* and *b* defined by (10). If *b>0* and $\frac{b^2}{4a} > 0$ then the system (2) is asymptotically stable. Applying the following simplifications from (10), we make the following estimates.

$$\begin{aligned} &\lambda_{max} M = m_{max} \\ &\lambda_{\max}\left(G * K^{-1} G\right) = g_{\max}^2 / k_{\max} \\ &\lambda_{\min} 2G = 2g_{\min} \\ &a = m_{\max} + \frac{1}{4} g_{\max}^2 k_{\max}, \quad b = 2g_{\max} \end{aligned} \tag{12}$$

Applying (12) on $\frac{b^2}{4a} > 0$

We have the following condition

$$2g_{\max} > 0, \quad 4k_{\min} g_{\max}^2 / \left(4m_{\max} k_{\min} + g_{\max}^2\right) > 0 \tag{13}$$

From (11), we choose an appropriate $\gamma > 0$ as $\gamma = \frac{b}{a}$

$$\gamma = 8g_{\max} k_{\min} / \left(4m_{\max} k_{\min} + g_{\max}^2\right) \tag{14}$$

Eqn. (13) is a simple sufficient condition for asymptotic stability of system (2) [13].

4 Response Bounds for Homogeneous Case

The homogeneous system (2) which is assumed to be stable is considered. The stability of the system implies there exists a value $\gamma > 0$ and a Lyapunov function V for a given solution *x(t)* . Thus, we have the following

$$\begin{aligned} &V = z(t)^* P z(t) \\ &V = x^*(t)\left(K - \frac{\gamma^2}{4} M\right) x(t) + \left(\dot{x}(t) + \frac{\gamma}{2} x(t)\right)^* M \left(\dot{x}(t) + \frac{\gamma}{2} x(t)\right) \le V_0 \end{aligned} \tag{15}$$

where V_0 is the initial energy given by the initial condition

$$V_0 = x^*(0)\left(K - \frac{\gamma^2}{4}M\right)x(0) + \left(\dot{x}(0) + \frac{\gamma}{2}x(0)\right)^* M\left(\dot{x}(0) + \frac{\gamma}{2}x(0t)\right)$$

We now establish the response bounds for the amplitude and velocity. To obtain a bound for the amplitude of *x(t)*, we estimate the first term of V.

$$0 \le \lambda_{\min}\left(K - \frac{\gamma^2}{4}M\right)x^*(t)x(t) \le x^*(t)\left(K - \frac{\gamma^2}{4}M\right)x(t) \tag{16}$$

From (17)

$$x^*(t)\left(K - \frac{\gamma^2}{4}M\right)x(t) \le V_0 \tag{17}$$

Therefore,

$$\lambda_{\min}\left(K - \frac{\gamma^2}{4}M\right)x^*(t)x(t) \le V_0 \tag{18}$$

But

$$x^*(t)x(t) = \|x(t)\|^2 \tag{19}$$

Applying (19) on (18) we have

$$\lambda_{\min}\left(K - \frac{\gamma^2}{4}M\right)\|x(t)\|^2 \le V_0$$

$$\|x(t)\| \le \sqrt{\frac{V_0}{\lambda_{\min}\left(K - \frac{\gamma^2}{4}M\right)}} \tag{20}$$

To obtain the tightest bound, we choose $\gamma = \frac{b}{2a}$. This choice seems to be advantageous in general. We obtain a bound for the velocity $\dot{x}(t)$ by estimation of the second term of V as follows.

$$\left(\dot{x}(t)+\frac{\gamma}{2}x(t)\right)^{*}M\left(\dot{x}(t)+\frac{\gamma}{2}x(t)\right)\geq\lambda_{\min}(M)\left(\dot{x}(t)+\frac{\gamma}{2}x(t)\right)^{*}\left(\dot{x}(t)+\frac{\gamma}{2}x(t)\right)$$
$$\geq\lambda_{\min}(M)\left\|x(t)+\frac{\gamma}{2}x(t)\right\|^{2}\geq\lambda_{\min}(M)\left(\|\dot{x}(t)\|+\frac{\gamma}{2}x(t)\right)^{2}$$
$$\geq\lambda_{\min}(M)\left(\|\dot{x}(t)\|-\frac{\gamma}{2}\|x(t)\|\right)^{2} \tag{21}$$

From (15)

$$\left(\dot{x}(t)+\frac{\gamma}{2}x(t)\right)^{*}M\left(\dot{x}(t)+\frac{\gamma}{2}x(t)\right)\leq V_0$$

Therefore it implies that

$$\lambda_{\min}(M)\left(\|\dot{x}(t)\|-\frac{\gamma}{2}\|x(t)\|\right)^{2}\leq V_0$$
$$\|\dot{x}(t)\|\leq\frac{\gamma}{2}\|x(t)\|+\sqrt{\frac{V_0}{\lambda_{\min}(M)}} \tag{22}$$

In addition to the estimates of the norms it is possible to find bounds for every individual co-ordinates. For a given quadratic form $V = z(t)^{*}Pz(t), P > 0$ and for a fixed value V, we can give an upper bound for the co-ordinate z_k as

$$|z_k|\leq\sqrt{VP_{kk}^{-1}} \tag{23}$$

where P_{kk}^{-1} is the kth diagonal element of the matrix P^{-1}.

Analogous to (20) the amplitude bound for $x_k(t)$ is

$$|x_k(t)|\leq\sqrt{V_0\left(K-\frac{\gamma^2}{4}M\right)_{kk}^{-1}} \tag{24}$$

A bound for $\dot{x}_k(t)$ can similarly be found from

$$\left|\dot{x}_k(t)+\frac{\gamma}{2}x_k(t)\right|\leq\sqrt{V_0M_{kk}^{-1}} \tag{25}$$

where M_{kk}^{-1} is the kth diagonal element of the inverse matrix M^{-1}.

It follows from (25) that

$$\left|\dot{x}_k(t)\right| + \frac{\gamma}{2}\left|x_k(t)\right| \le \sqrt{V_0 M_{kk}^{-1}}$$

But

$$\left|\dot{x}_k(t)\right| - \frac{\gamma}{2}\left|x_k(t)\right| \le \left|\dot{x}_k(t)\right| + \frac{\gamma}{2}\left|x_k(t)\right| \le \sqrt{V_0 M_{kk}^{-1}}$$

Therefore

$$\left|\dot{x}_k(t)\right| \le \frac{\gamma}{2}\left|x_k(t)\right| + \sqrt{V_0 M_{kk}^{-1}} \tag{26}$$

5 Response Bounds for the Inhomogeneous Case

Consider the inhomogeneous system

$$M\ddot{x} + G\dot{x} + Kx = f(t) \tag{27}$$

which we again assume is stable in accordance with Theorem 1. The response bounds for a solution x(t) of (27) satisfying the given initial conditions x(0) and $\dot{x}(0)$ can be established. For a non-transient excitation f(t) it is normally easy to find a particular solution $x_{part}(t)$ and its corresponding state and velocity bounds. We then define a solution $x_h(t) = x(t) - x_{part}(t)$ to the homogeneous system of (27) with the initial conditions, $x_h(0) = x(0) - x_{part}(0)$ and $\dot{x}_h(0) = \dot{x}(0) - \dot{x}_{part}(0)$. For $x_h(t)$ we thus have

$$V_{0,h} = x_h^*(0)\left(K + -\frac{\gamma^2}{4}M\right)x_h(0) + \left(\dot{x}_h(0) + \frac{\gamma}{2}x_h(0)\right)^* M\left(\dot{x}_h(0) + \frac{\gamma}{2}x_h(0)\right)$$

where $V_{0,h}$ is the initial energy condition of the homogenous system of (27).

Using (20) and (24) the earlier results for the response bounds of *x(t)* and $x_k(t)$, we have

$$\left\|x(t)\right\| \le \sqrt{\frac{V_{0,h}}{\lambda_{\min}\left(K - \frac{\gamma^2}{4}M\right)}} + \left\|x_{part}(t)\right\|$$

and

$$\left|x_k(t)\right| \le \sqrt{V_{0,h}\left(K - \frac{\gamma^2}{4}M\right)^{-1}} + \left|x_{part}(t)\right|$$

Similarly, using (19) and (26), we obtain the following response bounds for $\dot{x}(t)$ and $\dot{x}_k(t)$ as

$$\|x(t)\| \le \frac{\gamma}{2}\|x(t)\| + \sqrt{\frac{V_{0,h}}{\lambda_{\min}(M)}} + \|\dot{x}_{part}(t)\|$$

and

$$|x_k(t)| \le \sqrt{V_{0,h}\left(K - \frac{\gamma^2}{4}M\right)^{-1}} + |x_{part}(t)|$$

For a transient excitation f(t), we can find a solution to (27) with the initial conditions *x(0) = 0* and $\dot{x}(0) = 0$ by calculating the convolution of the impulse response matrix $\phi(t)$ and *f(t).* The solution of (27) is as follows:

$$x(t) = \int_0^t \phi(t-\tau) f(\tau) d\tau \tag{28}$$

The impulse response matrix ϕ(t) satisfies

$$M\ddot{\phi} + G\dot{\phi} + K\phi = 0, \quad \phi(0) = 0, \quad M\dot{\phi}(0) = 1$$

where *I* is the identity matrix.

We now assume that the excitation vector f(t) has the form

$$F(t) = u\psi(t)$$

where u is a constant vector and $\psi(t)$ is a scalar function subjected to

$$P = \int_0^\infty |\psi(t)| dt < \alpha$$

To obtain bounds of solution *x(t)* given by (28), we have to estimate the solution to the homogeneous equation $\varphi(t) = \phi(t)u$ which satisfies the initial conditions $\varphi(0) = 0$ and $\dot{\varphi}(0) = \dot{\phi}(0)u = M^{-1}u$, and therefore $V_{0,h} = u^* M^{-1} u$. This leads to the following estimate of the 2-norm of the solution *x(t)*.

$$\|x(t)\| \le \sqrt{\frac{u^* M^{-1} u}{\lambda_{\min}\left(K - \frac{\gamma^2}{4}M\right)}} P \tag{29}$$

By using (26), we can also obtain an estimate for the co-ordinate $x_k(t)$ of the solution *x(t)* as follows:

$$|x_k(t)| \le \sqrt{u^* M^{-1} u \left(K - \frac{\gamma^2}{2}M\right)^{-1}_{kk}} P \tag{30}$$

6 Illustrations

Example 1:

To illustrate the formulas for the response bounds of the homogeneous system (2), let us consider the 2x2 system described by

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_i \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{31}$$

We obtain the values of the constants a and b as defined in (11) as follows:

$$a = \lambda_{\max}(M + \frac{1}{4}G^* K^{-1} G) = \lambda_{\max}(4.3463, 1.6537) = 4.3463$$
$$b = \lambda_{\min} 2G = \lambda_{\min}(4i, -4i) = -4i$$
$$\frac{b^2}{4a} = \frac{(-4i)^2}{4(4.3463)} = \frac{-16}{17.3852} = -0.9203 \ngtr 0$$

Therefore the system (31) is unstable.

Example 2:

The Example 1 above shows the case where the system is unstable because of non-satisfaction of the conditions in theorem 1. In this example, we illustrate a system which is stable. Consider the 2x2 system described by

$$\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_i \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. = f(t) \tag{32}$$

The constants a and b defined in (11) are as follows:

$$a = \lambda_{\max}(M + \frac{1}{4}G^* K^{-1} G) = \lambda_{\max}(4,7) = 7$$
$$b = \lambda_{\min}(2G) = \lambda_{\min}(6,10) = 6$$
$$Thus\ \frac{b^2}{4a} = \frac{36}{28} = 1.2857 > 0$$
$$\text{b} > 0 \text{ and } \frac{b^2}{4a} > 0.$$

System (32) satisfies the conditions of Thm1 and is therefore stable.

7 Conclusion

Gyroscopic systems are important mathematical models for many science and engineering systems. The stability or instability of gyroscopic systems plays an important role in many problem areas. Lyapunov direct method have been used to analyse the stability or otherwise of gyroscopic system. The response bounds for displacements and velocities both in the homogeneous and inhomogeneous cases have been obtained. A novel stability theorem has been developed for determining the stability or otherwise of gyroscopic systems. Examples have been given to illustrate the efficacy of the results obtained.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Greenlee WM. Lyapunov stability of linear gyroscopic systems. Utilitas Matheamtica. 1975;8:225-231.

[2] Connell GM. Asymptotic stability of second-order linear systems with semidefinite damping; 1969.

[3] Kirillov ON. On the stability of nonconservative system with small dissipation. Journal of Mathematical Sciences. 2007;145:5260–5270.

[4] Seiranyan AP, Kirillov ON. Effect of small dissipative and gyroscopic forces on the stability of nonconservative system. Doklady Physics. 2003;48(12):679-684.

[5] Schichlen W, Hu B. Amplitude bounds of linear vibration responses. ZAMM (Z. Angew. Math. Mech). 1996;76:453–454.

[6] Kliem W, Pommer C. Stability and response bounds of non-conservative linear systems. Archives of Applied Mechanics. 2004;73:627-637.

[7] Schichlen W, Hu B. Amplitude bounds of linear free vibrations. ASME J. Appl. Mech. 1995;6:231–233.

[8] Kliem W, Pommer C, Stoustrup J. Stability of rotor systems: A complex modeling approach. ZAMP (Z. Angew. Math. Phys). 1998;49:644-655.

[9] Lancaster P. Stability of linear gyroscopic systems: A review. Linear Algebra and Its Applications. 2013;439(3):689-706.

[10] Bulatovic RM. Condition for instability of conservative gyroscopic systems. Theoretical and Applied Mechanics. 2001;26:127-133.

[11] Muller PC. Stabilitat and matrizen. Springer-Verlag, Berlin Heideelberg, New York; 1977.

[12] Horn R, Johnson CA. Matrix analysis. Cambridge University Press; 1985.

[13] Frik M. Stabilitat nichtkonservativer linearer. ZAMM (Z. Angew. Math. Mech). 1972;52:T47-T49.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://sciencedomain.org/review-history/20229