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A Refinement of the Asymptotic Growth Bounds for the Vlasov Poisson System with a Point Charge

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Short Research Article

Abstract

In [1] proved that global existence and uniqueness of a classical solution to the three dimensional Vlasov-Poisson system in presence of point charges in case of repulsive interaction. Authors in [2] were the first to establish a growth bound on the size of the velocity support of the phase space density. This paper improves it further.

Keywords: Vlasov-Poisson system; point charge; velocity growth; classical solution.

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1 Introduction

In this paper we study the time evolution of a positive plasma interacting with a positive point charge in three dimensional case. Let $f(t,x,v)$ be the one particle distributional function of the plasma, then the macroscopic density of the plasma is defined by $\rho(t,x) = \int_{R^3} f dv$. Particles in

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the plasma are subjected to two forces: one is the self-induced electrostatic force $E(t, x)$ and the other is the repulsive force $F(t, x)$ generated by the positive point charge. Then the three dimension Vlasov-Poisson system with a positive point charge writes:

$$\begin{aligned} &\partial_t f + v \cdot \nabla_x f + (E + F)\nabla_v f = 0, \\ &E(t, x) = \int_{R^3} \frac{(x - y)}{|x - y|^3} \rho(t, y)dy, \\ &F(t, x) = \frac{x - \xi(t)}{|x - \xi(t)|^3}, \quad f(0, x, v) = f_0(x, v), \\ &\dot{\xi}(t) = \eta(t), \quad \dot{\eta}(t) = E(t, \xi(t)), \\ &(\xi(0), \eta(0)) = (\xi_0, \eta_0), \end{aligned} \tag{1.1}$$

where $f_0(x, v) \geqslant 0$ is the initial microscopic density of the plasma, which is assumed to be known. In which $\xi_0 \in \mathbb{R}^3$ and $\eta_0 \in \mathbb{R}^3$ are the initial position and velocity of the point charge, respectively. In absence of the charge, the system (1.1) reduces to the well-known Vlasov-Poisson equation, which has been widely investigated about existence and uniqueness of classical solutions in the last years. In two and three dimensional problem was solved in the nineties in [3, 4, 5, 6]: In the framework of compactly supported solutions a natural way to quantify the number of fast moving particles consists in looking at the size of the support of the velocity variable

$$R(t) = \sup\{|v| : \exists\ x\ \in \mathbb{R}^3,\ f(t, x, v) \neq 0\}.$$

Once, asymptotic growth bounds on $R(t)$ are established, see [7, 8, 9, 10] for several different estimates, so far the best results are as follows :

$$R(t) \leq C(1 + t)^{\frac{2}{3}} \ln^{\frac{11}{21}}(2 + t),\ t \geq 0,$$

in the attractive case ($\gamma = -1$) [11]. For any given $\epsilon > 0$ there exist a positive constant c, such that

$$R(t) \leq C_\epsilon(1 + t)^{\frac{2}{11}+\epsilon},\ t \geq 0,$$

in the repulsive case ($\gamma = +1$) [12]. [13, 14, 15, 16] studied and proved global existence and unique of classical solution to the Vlasov-Poisson system with point charge, the analysis of these papers relies on an essential tool: the microscopic energy $h(t, x, v)$, which is defined by

$$h(t, x, v) = \frac{|v - \eta(t)|^2}{2} + \frac{1}{|x - \xi(t)|} + G, \tag{1.2}$$

where G is a suitable constant. So far, the first upper bound of exponential type for (1.1) was obtained in [1]

$$R(t) \leq C(C + Q_0)\exp(C(1 + t)), t \geq 0,$$

which was improved in [8], namely

$$R(t) \leq CQ_0(1 + t)^{\frac{15}{2}}, t \geq 0,$$

where C and Q_0 are generic positive constants depending only upon initial data f_0. This paper combines the methods in the papers [1] and [10] to refine a bit that estimate and obtain:

Theorem 1.1. *Let $f(t, x, v)$ be a classical solution of the Vlasov Poisson system* (1.1) *with a non negative initial datum $f_0(x, v) \in C^1_c(\mathbb{R}^6)$, $(\xi_0, \eta_0) \in \mathbb{R}^3 \times \mathbb{R}^3$ and assume $\min\{|x - \xi_0| \mid (x, v) \in suppf_0(x, v)\} > 0$. Then there exists a constant $C > 0$ such that*

$$R(t) \leq CQ_0(1 + t)^{\frac{21}{5}}, \quad t \geq 0, \tag{1.3}$$

where $Q_0 = \sup\{\sqrt{h}(0, x, v) \mid (x, v) \in\ suppf_0\}$.

In this paper, the letter C is a generic constant depending only on f_0, and $\|.\|_p$ always denote the norm of the space $L^p(\mathbb{R}^3)$ for $1 \leq p \leq \infty$.

2 Preliminaries and Dynamical Estimates

We use this section to fix the notations, to recall some preliminary result (see [1]and references quoted therein). In the remainder of this article for any fix $T > 0$, we say that (f, ξ) is a unique classical solution of (1.1) on $[0, T]$, with initial condition $(f_0, (\xi_0, \eta_0))$ satisfying the assumptions in Theorem 1.1 such that $f_0 \in C_c{}^1(\mathbb{R}^6)$, $f \in C_c{}^1([0, T]\times\mathbb{R}^3\times\mathbb{R}^3)$ and $\xi \in C^2([0, T])$, we note:

i)For any fixed $s \in \mathbb{R}_+$, denote $(X(s; 0), V(s; 0)) = (X(s; 0, x, v), V(s; 0, x, v))$ is the solution of characteristic system:

$$\begin{cases} \dot{X}(s;0) = V(s;0), \\ \dot{V}(s;0) = E(s, X(s;0)) + \frac{X(s;0)-\xi(s)}{|X(s;0)-\xi(s)|^3}, \\ (X(0;0), V(0;0)) = (x, v), \end{cases} \tag{2.1}$$

where $(x, v) \in \mathbb{R}^3\backslash\{\xi_0\} \times \mathbb{R}^3$.

ii)For any fixed $s \in \mathbb{R}_+$, the map $(x, v) \mapsto (X(s; 0), V(s; 0))$ is a measure preserving C^1 bijection from $(\mathbb{R}^3\backslash\{\xi_0\})\times\mathbb{R}^3$ onto $(\mathbb{R}^3\backslash\{\xi(s)\})\times\mathbb{R}^3$, since we know by classical theory of ordinary differential equations that $(X(s; 0), V(s; 0)) \in C^1(\mathbb{R}_+)$ for any fixed (x, v), and

$$f(t, x, v) = f_0(x, v).$$

Consequently for any $p \in [1, \infty]$ the L^p norm $\|f(t)\|_p$ is preserved

$$\|f(t)\|_p = \|f_0\|_p, \quad \text{for any } t \geq 0.$$

Furthermore, the energy of the system (1.1) at time $t \geq 0$ is defined by

$$H(t) = \frac{1}{2}\int_{\mathbb{R}^6} |v|^2 f(t, x, v)dxdv + \frac{1}{2}|\eta(t)|^2 + \frac{1}{2}\int_{\mathbb{R}^6} \frac{\rho(t, x)\rho(t, y)}{|x-y|}dxdy + \int_{\mathbb{R}^3} \frac{\rho(t, x)}{|x-\xi(t)|}dx$$

is also preserved namely: $H(t) = H(0)$ for $t > 0$. Throughout this work we will use the following lemma established in [1].

Lemma 2.1. *There exists a constant $K > 0$ depending on $H(0)$ and $\|f_0\|_\infty$ for which*

$$\int_{\mathbb{R}^3} \frac{\rho_R(x')}{|x-x'|^2}dx' \leq KR^{\frac{4}{3}}, \tag{2.2}$$

where $\rho_R(x) = \int_{|v|\leq R} f(t, x, v)dv$ and $R > 0$. Without loss of generality, we may assume that $K \geq \max\{H(0), 1\}$. Due to the assumptions on the initial datum $f_0(x, v)$, one can easily deduce that $H(t) = H(0) \leq C < \infty$. In this paper we chose $G = K$ in (2.1) where K is a large constant defined in Lemma 2.1. Then we have

$$|v| \leq |v-\eta| + |\eta| \leq \sqrt{2h}(t, x, v) + \sqrt{2H(0)} \leq 2\sqrt{2h}(t, x, v), \tag{2.3}$$

thus combining the initial condition $f_0(x, v)$ with $\|f(t)\|_p$, we get that for $t \geq 0$

$$\begin{aligned} & \int_{\mathbb{R}^6} h(t, x, v)f(t, x, v)dxdv \\ \leq \quad & \int_{\mathbb{R}^6} |v|^2 f(t, x, v)dxdv + (|\eta|^2 + K)\|f_0\|_{L^1(\mathbb{R}^6)} + \int_{\mathbb{R}^3} \frac{\rho(t, x)}{|x-\xi(t)|}dx \\ \leq \quad & 3H(0) + (2H(0) + K)\|f_0\|_{L^1} \leq C. \end{aligned} \tag{2.4}$$

Now, we define

$$P(t) = \sup\{\sqrt{h}(s, X(s; 0), V(s; 0)) \mid s \in [0, t],\ (x, v) \in \text{supp} f_0\}.$$

And for $0 < \delta < t$ define

$$Q(t, \delta) = \sup\{\sqrt{h}(s, X(s; t-\delta), V(s; t-\delta)) \mid s \in [t-\delta, t],\ (x, v) \in \mathrm{supp} f(t-\delta)\}.$$

Next, differentiating along trajectories (2.1) with (2.3), we find

$$|\frac{d}{ds} h(s, X(s; 0), V(s; 0))| = |\frac{1}{2\sqrt{h}}(V(s; 0) - \eta(s)) \cdot (E(s, X(s; 0)) - E(s, \xi(s)))|,$$

from which

$$|\frac{d}{ds}\sqrt{h}(s, X(s; 0), V(s; 0))| \leq |E(s, X(s; 0))| + |E(s, \xi(s))|. \tag{2.5}$$

As explained earlier, the method relies on a suitable splitting of $[0, t]$ into small intervals. For a given $t > 0$, we split the interval $[0, t]$ as follows $(0, t] = \bigcup_{i=1}^{n-1}(t_{i-1}, t_i]$. More precisely we set $\delta(t) = (t_{i-1} - t_i] = \frac{1}{8KP(t)^{\frac{16}{21}}}$. Hence, from (2.5) we have that

$$\begin{aligned}\sqrt{h}(t, X(t; t-\delta), V(t; t-\delta)) \quad &\leq \quad \sqrt{h}(t-\delta, x, v)\\ &+ \quad \int_{t-\delta}^{t}(|E(s, X(s; t-\delta))| + |E(s, \xi(s))|)ds, \end{aligned}\tag{2.6}$$

$$\sqrt{h}(t, X(t; 0), V(t; 0)) \leq \sqrt{h}(0, x, v) + \int_0^t(|E(s, X(s; 0))| + |E(s, \xi(s))|)ds. \tag{2.7}$$

Now, we need to control the electric fields $|E(s, X(s; t-\delta))|$ and $|E(s, \xi(s))|$. One can control the quantity $|E(s, \xi(s))|$ by the following proposition: where we will use a variance substitute $\bar{y} = X^*(s) = X(s; t-\delta(t), y, w)$, $\bar{w} = V^*(s) = V(s; t-\delta(t), y, w)$ and utilized the property that $f(t, x, v)$ is a constant along the characteristic flow, in the following propositions.

Proposition 2.2. *: Let (f, ξ) be a classical solution of (1.1) on $[0, T]$. we have*

$$\int_{t-\delta(t)}^{t}|E(s, \xi(s))|ds \leq CP(t)^{\frac{16}{21}}\delta(t). \tag{2.8}$$

Proof:

$$\begin{aligned}\int_{t-\delta(t)}^{t}|E(s, \xi(s))|ds \quad &\leq \quad \int_{t-\delta(t)}^{t}\int_{R^6}\frac{f(s, \bar{y}, \bar{w})}{|\bar{y} - \xi(s)|^2}d\bar{y}d\bar{w}ds\\ &\leq \quad \int_{t-\delta(t)}^{t}\int_{R^6}\frac{f(t-\delta(t), y, w)}{|X^*(s) - \xi(s)|^2}dydwds,\end{aligned}$$

Setting $U(s) = |X^*(s) - \xi(s)|$, then

$$\begin{aligned}\dot{U}(s) \quad &= \quad \frac{X^*(s) - \xi(s)}{|X^*(s) - \xi(s)|} \cdot (V^*(s) - \eta(s)),\\ \ddot{U}(s) \quad &= \quad \frac{|V^*(s) - \eta(s)|^2}{|X^*(s) - \xi(s)|} + \frac{1}{U(s)^2} + \frac{X^*(s) - \xi(s)}{|X^*(s) - \xi(s)|} \cdot (E(X^*(s), s) - E(\xi(s)), s)\\ &\quad - \frac{[(X^*(s) - \xi(s)) \cdot (V^*(s) - \eta(s))]^2}{|X^*(s) - \xi(s)|^3}.\end{aligned}$$

Using Lemma 2.1 and definition of $P(t)$ (see [15]), we obtain

$$\ddot{U}(s) \geq \frac{1}{U(s)^2} - 2KP(t)^{\frac{4}{3}},$$

therefore,

$$\begin{aligned}\int_{t-\delta(t)}^{t}\frac{1}{U(s)^2}ds &\leq \dot{U}(t)-\dot{U}(t-\delta(t))+2KP(t)^{\frac{4}{3}}\delta(t)\\ &\leq |V^*(t)-\eta(t)|+|V^*(t-\delta(t))-\eta(t-\delta(t))|+2KP(t)^{\frac{4}{3}}\delta(t)\\ &\leq CP(t)+\frac{1}{4}P(t)^{\frac{13}{20}}\leq CP(t).\end{aligned}$$

Now choose

$$P=P(t)^{\frac{4}{7}},$$

with using the last inequality, we have

$$\begin{aligned}\int_{t-\delta(t)}^{t}|E(s,\xi(s))|ds &\leq \int_{t-\delta(t)}^{t}\int_{R^6}\frac{f(t-\delta(t),y,w)}{|X^*(s)-\xi(s)|^2}dydwds\\ &\leq \int_{t-\delta(t)}^{t}\int_{\{\sqrt{h}(s,X^*(s),V^*(s))\leq P\}}\frac{f(t-\delta(t),y,w)}{|X^*(s)-\xi(s)|^2}dydwds\\ &+ \int_{t-\delta(t)}^{t}\int_{\{\sqrt{h}(s,X^*(s),V^*(s))\geq P\}}\frac{f(t-\delta(t),y,w)}{|X^*(s)-\xi(s)|^2}dydwds\\ &\leq \int_{t-\delta(t)}^{t}\int_{\{\sqrt{h}(s,X^*(s),V^*(s))\leq P\}}\frac{f(t-\delta(t),y,w)}{|X^*(s)-\xi(s)|^2}dydwds\\ &+ \int_{\{\sqrt{h}(s,X^*(s),V^*(s))\geq P\}}f(t-\delta(t),y,w)\int_{t-\delta(t)}^{t}\frac{ds}{|X^*(s)-\xi(s)|^2}dydw\\ &\leq CP^{\frac{4}{3}}\delta(t)+CP(t)P^{-2}\leq CP(t)^{\frac{16}{21}}\delta(t)+CP(t)^{-\frac{1}{7}}\\ &\leq CP(t)^{\frac{16}{21}}\delta(t)+CP(t)^{\frac{13}{21}}\delta(t)\leq C\delta(t)P(t)^{\frac{16}{21}}.\end{aligned}$$

where we have used measure preserving of characteristic flow and (2.4) in the above inequality to get that

$$\begin{aligned}\int_{R^6}f(t-\delta(t),y,w)dydw &= \int_{R^6}\frac{h(s,X^*(s),V^*(s))}{h(s,X^*(s),V^*(s))}f(t-\delta(t),y,w)dydw\\ &\leq P^{-2}\int_{R^6}h(s,\bar{y},\bar{w})f(t-\delta(t),y,w)dydw\\ &\leq CP^{-2}.\end{aligned}\tag{2.9}$$

□

Proposition 2.3. *: For any $t>0$, $\delta(t)=\frac{1}{8KP(t)^{\frac{16}{21}}}<t$ and $Q(t,\delta)\geq(128K)^2$, we have*

$$\int_{t-\delta(t)}^{t}|E(s,X(s;t-\delta(t))|ds\leq C\delta(t)P(t)^{\frac{16}{21}}.\tag{2.10}$$

Proof : Firstly we choose $L = P^{\frac{-2}{3}}$ and $r(s,v)) = |v - V^*(s)|^{-1}|v|^2$, and define

$$\begin{aligned}
S_1 &= \{(s,y,w) \in (t-\delta(t),t) \times \mathrm{supp} f(t-\delta(t)) \mid \sqrt{h}(s, X^*(s), V^*(s)) \leq P\},\\
S_2 &= \{(s,y,w) \in (t-\delta(t),t) \times \mathrm{supp} f(t-\delta(t)) \mid \sqrt{h}(s, X^*(s), V^*(s)) \geq P,\\
&\quad |X(s; t-\delta(t)) - X^*(s)| \leq r(s,v)\},\\
S_3 &= \{(s,y,w) \in (t-\delta(t),t) \times \mathrm{supp} f(t-\delta(t)) \mid \sqrt{h}(s, X^*(s), V^*(s)) \geq P,\\
&\quad |X(s; t-\delta(t)) - X^*(s)| \geq r(s,v), |X^*(s) - \xi(s)| > L, |X(s;t-\delta(t)) - \xi(s)| > 2L\},\\
S_4 &= \{(s,y,w) \in (t-\delta(t),t) \times \mathrm{supp} f(t-\delta(t)) \mid \sqrt{h}(s, X^*(s), V^*(s)) \geq P,\\
&\quad |X(s; t-\delta(t)) - X^*(s)| \geq r(s,v), |X^*(s) - \xi(s)| < L, |X(s;t-\delta(t)) - \xi(s)| > 2L\},\\
S_5 &= \{(s,y,w) \in (t-\delta(t),t) \times \mathrm{supp} f(t-\delta(t)) \mid \sqrt{h}(s, X^*(s), V^*(s)) \geq P,\\
&\quad |X(s; t-\delta(t)) - X^*(s)| \geq r(s,v), |X^*(s) - \xi(s)| > L, |X(s;t-\delta(t)) - \xi(s)| < 2L\}.
\end{aligned}$$

We denote

$$\begin{aligned}
I =: \int_{t-\delta(t)}^{t} |E(s, X(s; t-\delta(t)))| ds &\leq \int_{t-\delta(t)}^{t} \int_{\mathbb{R}^6} \frac{f(s, \bar{y}, \bar{w})}{|X(s; t-\delta(t)) - \bar{y}|^2} d\bar{y} d\bar{w} ds\\
&\leq \int_{t-\delta(t)}^{t} \int_{\mathbb{R}^6} \frac{f(t-\delta(t), y, w)}{|X(s; t-\delta(t)) - X^*(s)|^2} dy dw ds
\end{aligned}$$

Let I_i denote the contribution of the set S_i to the above integral. The integral I_1 is easily estimated, by using Lemma 2.1

$$\begin{aligned}
I_1 &\leq \int_{t-\delta(t)}^{t} \int_{\mathbb{R}^3 \times \{|\bar{w}| < 4P\}} \frac{f(s, \bar{y}, \bar{w})}{|X(s; t-\delta(t)) - \bar{y}|^2} d\bar{y} d\bar{w} ds\\
&\leq K\delta(t)(4P)^{\frac{4}{3}} \leq C\delta(t)P(t)^{\frac{16}{21}}. \qquad (2.11)
\end{aligned}$$

To estimate I_2 pick (x,v) such that $|v| = P(t)$, if $\delta \leq (t, \frac{P(t)}{2})$ then $|v| \leq C|V^*|$ and $P(s) \leq CP(t)$, therefore for any $s \in (t-\delta, t)$ we can estimate I_2 by integrating in the space variable first

$$\begin{aligned}
I_2 &\leq \int_{S_1} \frac{f(t-\delta(t), y, w)}{|X(s; t-\delta(t)) - X^*(s)|^2} dy dw ds \leq \int_{t-\delta}^{t} \int \frac{1}{|v|^2 |v - V^*(s)|} dv ds\\
&\leq C \int_{t-\delta}^{t} (1 + \ln \frac{P(s)}{|V^*(s)|}) ds \leq C\delta(t) \leq C\delta(t)P(t)^{\frac{16}{21}}. \qquad (2.12)
\end{aligned}$$

To estimate I_3 we define

$$\triangle(t,p) = sup\{\delta \in (0,t) | \forall (x,v) \in supp f_0 \int_{t-\delta(t)}^{t} |E(s, X(s,0,x,v))| ds \leq 4p\}$$

and we assume $|V(t; t-\delta(t)) - V^*(t)| \geq 4p$. The main estimate in the process is given in the lemma below

Lemma 2.4. *: Let $(x,v) \in supp f(t-\delta)$ and $\delta \in (0,t)$ verifying*

$$\delta \leq \triangle(t, \frac{1}{5}|V(t; t-\delta) - V^*(t)|). \qquad (2.13)$$

Suppose there exists $\lambda > 0$ such that

$$r(s, V(s; t-\delta(t)) \geq \lambda r(t, V(t; t-\delta(t))). \qquad (2.14)$$

For all $s \in [t-\delta, t]$, then

$$\begin{aligned}
&\int_{t-\delta(t)}^{t} |X(s; t-\delta(t)) - X^*(s)|^{-2} 1_{s_3}(s, X(s), V(s)) ds\\
&\qquad \leq \frac{C}{|V(t; t-\delta) - V^*(t)| r(t, V(t; t-\delta(t)))}. \qquad (2.15)
\end{aligned}$$

Proof : Let $Z(s) = X(s; t - \delta(t)) - X^*(s)$, we note

$$\begin{aligned}\dot{Z}(s) &= V(s; t - \delta(t)) - V^*(s)\\ \ddot{Z}(s) &= E(s, X(s; t - \delta(t))) - E(s, X^*(s))\\ &+ \frac{X(s; t - \delta(t)) - \xi(s)}{|X(s; t - \delta(t)) - \xi(s)|^3} - \frac{X^*(s) - \xi(s)}{|X^*(s) - \xi(s)|^3},\end{aligned}$$

for any s, suppose s_0 minimizes $|Z(s)|^2$, it comes $|Z(s)| \geq |Z(s_0) + (s - s_0)\dot{Z}(s_0)| - |\int_{s_0}^{s}(s - u)|\ddot{Z}(u)|du|$. Besides, if $s = s_0$ minimizes $|Z(s)|^2$ when $s \in [t - \delta(t), t]$ then $(s - s_0)Z(s_0)\dot{Z}(s_0) \geq 0$, therefore

$$\begin{aligned}|Z(s)| &\geq |(s - s_0)\dot{Z}(s_0)| - |\int_{s_0}^{s}(s - u)|\ddot{Z}(u)|du| \geq |(s - s_0)|(|\dot{Z}(s_0)| - |\int_{s_0}^{s}|\ddot{Z}(u)|du|)\\ &\geq |(s - s_0)|(|\dot{Z}(s_0)| - \frac{2}{5}|\dot{Z}(t)|).\end{aligned}$$

Since $\delta \leq \triangle(t, \frac{1}{5}|\dot{Z}(t)|)$, we have $|\dot{Z}(s_0) - \dot{Z}(t)| \leq \frac{2}{5}|\dot{Z}(t)|$. Hence

$$|X(s; t - \delta(t)) - X^*(s)| = |Z(s)| \geq \frac{1}{5}|s - s_0||\dot{Z}(t)| = \frac{1}{5}|s - s_0|V(t; t - \delta(t)) - V^*(t)|.$$

$$|Z(s)| \geq r(s, V(s; t - \delta)) \geq \lambda r(t, V(t; t - \delta(t)))$$

For any $s \in (t - \delta(t), t)$ and $(s, y, w) \in S_3$. Setting $\phi(u) = \min(u^{-2}, r^{-2})$, we have

$$\begin{aligned}\int_{t-\delta(t)}^{t} \frac{1_{s_3}(s, X(s), V(s))}{|X(s; t - \delta(t)) - X^*(s)|^2} ds &\leq \int_{-\infty}^{\infty} \phi(\frac{1}{5}|s - s_0|V(t; t - \delta(t)) - V^*(t)|)ds\\ &\leq \frac{C}{|V(t; t - \delta) - V^*(t)|r(t, V(t; t - \delta(t))}.\end{aligned}$$

This is the desired inequality . □

If $(s, X(s), V(s)) \notin S_3$, for any $s \in [t - \delta(t), t]$ the above estimate is verified. Other wise, there exist $s^* \in [t - \delta(t), t]$ such that $(s^*, X(s^*), V(s^*)) \in S_3$ and $\min(|V(s^*; t - \delta(t))|, |V(s^*; t - \delta(t)) - V^*(s^*)|) \geq 4p$. But then

$$\delta \leq \triangle(t, P) \leq \triangle(t, \frac{|V(s^*; t - \delta(t)) - V^*(s^*)|}{4}) \leq \triangle(t, \frac{1}{2}|V(t; t - \delta(t)) - V^*(t)|)$$

and assumption (2.13) holds true. Similarly we find $\delta \leq \triangle(t, \frac{1}{3}|V((t; t - \delta(t))|)$ so that $\delta(t) \leq \min(\triangle(t, \frac{1}{3}|V(t; t - \delta(t))|), \triangle(t, \frac{1}{2}|V(t; t - \delta(t)) - V^*(t)|))$. This gives in true $|V(s, t - \delta(t))| \leq \frac{3}{4}|V(t; t - \delta(t))|$ and $|V(s, t - \delta(t)) - V^*(s)| \leq 2|V(t; t - \delta(t)) - V^*(t)|$. In view of the definition of $r(s, v)$ the assumption (2.14) is satisfied. Thus we may use the lemma and (2.4), integrating yields

$$\begin{aligned}I_3 \quad &\leq \quad \int\int f(t - \delta(t), y, w) \int_{t-\delta(t)}^{t} |X(s; t - \delta(t)) - X^*(s)|^{-2}\\ 1_{s_3} \quad &(s, X(s; t - \delta(t)), V(s; t - \delta(t)))ds \leq C \leq C\delta(t)P^{\frac{16}{21}}(t). \qquad (2.16)\end{aligned}$$

On the other hand, if $|V(t; t - \delta(t)) - V^*(t)| \leq 4p$ by (2.1) we obtain that

$$\begin{aligned}\dot{V}(s; t - \delta(t)) - \dot{V}^*(s) &= E(s, X(s; t - \delta(t))) - E(s, X^*(s))\\ &+ \frac{X(s; t - \delta(t)) - \xi(s)}{|X(s; t - \delta(t)) - \xi(s)|^3} - \frac{X^*(s) - \xi(s)}{|X^*(s) - \xi(s)|^3},\end{aligned}$$

and

$$\frac{d}{ds}|V(s;t-\delta(t))-V^*(s)| \quad = \quad \frac{(V(s;t-\delta(t))-V^*(s))\cdot(\dot{V}(s;t-\delta(t))-\dot{V}^*(s))}{|V(s;t-\delta(t))-V^*(s)|},$$

by lemma 2.1 and definition of L, we infer

$\frac{d}{ds}|V(s;t-\delta(t))-V^*(s)| \leq 2(kP^{\frac{4}{3}}(t)+L^{-2}) \leq 2(1+k)P^{\frac{4}{3}}(t)$, therefore

$$\begin{aligned} |V(s;t-\delta(t))-V^*(s)| &\leq |V(t;t-\delta(t))-V^*(t)| + |\int_s^t \frac{d}{ds}|V(\tau;t-\delta(t))-V^*(\tau)|d\tau| \\ &\leq 4P(t)^{\frac{4}{7}} + 2(1+K)P(t)^{\frac{4}{3}}\delta(t) \leq \frac{9}{2}P(t)^{\frac{4}{7}}, \end{aligned}$$

we obtain

$$I_3 \leq C\delta(t)(\frac{9}{2}P^{\frac{4}{7}}(t))^{\frac{4}{3}} \leq C\delta(t)P^{\frac{16}{21}}(t). \tag{2.17}$$

The contribution of S_4, in the first, basically our aim is to show that the time spent by trajectory in the protection sphere $B(\xi(t),L)$ is very small, for proving this we apple the virial theorem argument introducing :$l(s)=\frac{1}{2}|X^*(s)-\xi(s)|^2$, differentiating we get

$$\begin{aligned} &\dot{l}(s) = (X^*(s)-\xi(s))\cdot(V^*(s)-\eta(s)), \\ &\ddot{l}(s) = |V^*(s)-\eta(s)|^2 + \frac{1}{|X^*(s)-\xi(s)|} + (X^*(s)-\xi(s))\cdot(E(s,X^*(s))-E(s,\xi(s))). \end{aligned}$$

Lemma 2.5. *: For $(x,v)\in suppf(t-\delta)$, then the set $N=\{s\in(t-\delta(t),t) \mid |X^*(s)-\xi(s)|<L\}$, is connected. Moreover,*

$$meas(N) \leq 2P^{\frac{-52}{21}}(t). \tag{2.18}$$

Proof: Let $s_0\in\bar{N}$ be administer for $l(s)$ by Lemma 2.1 and $\dot{l}(s)$ we have for $s\in[s_0,t)$

$$\begin{aligned} |\ddot{l}(s)| &\geq h(s,X^*(s),V^*(s)) - K - |X^*(s)-\xi(s)||E(s,X^*(s))-E(s,\xi(s))| \\ &\geq P^2 - K - 2KLP(t)^{\frac{4}{3}} \geq P(t)^{\frac{8}{7}} - \frac{1}{16}P(t)^{\frac{1}{2}} - \frac{1}{8}P(t)^{\frac{7}{6}} \geq \frac{1}{2}P(t)^{\frac{8}{7}}. \end{aligned} \tag{2.19}$$

Consider now $(s_1,s_2)\subset N$ is the maximal connected component containing s_0. If $s_0\in[s_1,s_2), \dot{l}(s)\geq 0$ (if $s_0=s_2$ we use the same argument via the time reversal), then

$$\dot{l}(s) \geq \dot{l}(s_0) + \int_{s_0}^s \ddot{l}(\tau)d\tau \geq \frac{1}{2}P^{\frac{8}{7}}(t)(s-s_0) \geq 0, \quad \forall s\in[s_0,t).$$

Since l is increasing from s_0 up to t, the trajectories can not reenter in the protection sphere once escaped. Therefore $N=(s_1,s_2)$ is connected. Next, integrating twice (2.19) in time and using $\dot{l}(s)\geq 0$, we get

$$\frac{1}{2}L^2 \geq l(s) = l(s_0) + \int_{s_0}^s \dot{l}(\tau)d\tau \geq \frac{1}{4}P(t)^{\frac{8}{7}}(s-s_0)^2, \quad s\in[s_0,s_2).$$

$$-\tfrac{1}{2}L^2 \leq -l(s) \leq l(s_0)-l(s) \leq -\tfrac{1}{4}P(t)^{\frac{4}{7}}(s_0-s)^2, \quad \forall s\in(s_1,s_0].$$

Similar estimate can be obtained when $s_0=s_1$ or $s_0=s_2$ so that $l(s)$ is a monotonic function and $N=(s_1,s_2)$ is connected, and (2.18) is proved . □

Next, to estimate the integral I_4, since

$$|X(s;t-\delta(t))-X^*(s)| \geq |X(s;t-\delta(t))-\xi(s)|-|X^*(s)-\xi(s)| \geq L.$$

Hence by Lemma 2.5 and (2.9), we have

$$\begin{aligned} I_4 &\leq L^{-2}\int_{t-\delta(t)}^{t} I_{\{|X^*(s)-\xi(s)|<L\}}(s)ds\int_{\mathbb{R}^6} f(t-\delta(t),y,w)dydw \\ &\leq C\delta(t)P(t)^{-\frac{88}{21}}. \end{aligned} \tag{2.20}$$

The contribution of S_5, let $s_0 \in [t-\delta, t]$ such that

$$\begin{aligned} & \sqrt{h}(s,X(s;t-\delta(t)),V(s;t-\delta(t))) \\ \geq\ & \sqrt{h}(s_0,X(s_0;t-\delta(t)),V(s_0;t-\delta(t))) \\ & -\int_{t-\delta(t)}^{t}(|E(s,X(s;t-\delta(t)))|+|E(s,\xi(s))|)ds \\ \geq\ & Q(t,\delta(t))-2KQ(t,\delta(t))^{\frac{4}{3}}\delta(t) = Q(t,\delta(t))-\frac{2KQ(t,\delta(t))^{\frac{4}{3}}}{8KP(t)^{\frac{16}{21}}} \\ \geq\ & Q(t,\delta(t))-\frac{Q(t,\delta(t))^{\frac{4}{3}}}{4Q(t,\delta(t))^{\frac{16}{21}}} = Q(t,\delta(t))-\frac{1}{4}Q(t,\delta(t))^{\frac{4}{3}} \geq \frac{1}{2}Q(t,\delta(t)). \end{aligned}$$

By similar way to the estimate of I_4, we suppose $l_1(s)=\frac{1}{2}|X(s;t-\delta(t))-\xi(s)|^2$, then

$$\begin{aligned} |\ddot{l_1}(s)| &\geq |V(s;t-\delta(t))-\eta(s)|^2+\frac{1}{|X(s;t-\delta(t))-\xi(s)|} \\ &- |X(s;t-\delta(t))-\xi(s)||E(s,X(s;t-\delta(t)))-E(s,\xi(s))| \\ &\geq h(s,X(s;t-\delta(t)),V(s;t-\delta(t)))-K- \\ &\quad |X(s;t-\delta(t))-\xi(s)||E(s,X(s;t-\delta(t)))-E(s,\xi(s))| \\ &\geq \frac{1}{4}Q(t,\delta(t))^2-K-4KL(2\sqrt{2}Q(t,\delta(t)))^{\frac{4}{3}} \\ &\geq \frac{1}{4}Q(t,\delta(t))^2-\frac{1}{16}Q(t,\delta(t))^{\frac{1}{2}}-\frac{16Q(t,\delta(t))^{\frac{4}{3}}Q(t,\delta(t))^{\frac{1}{2}}}{128P(t)^{\frac{2}{3}}} \\ &\geq \frac{1}{4}Q(t,\delta(t))^2-\frac{1}{16}Q(t,\delta(t))^{\frac{1}{2}}-\frac{1}{8}Q(t,\delta(t))^{\frac{7}{6}} \geq \frac{1}{16}Q(t,\delta(t))^2. \end{aligned}$$

Finally, we infer that

$$\begin{aligned} J_5 &\leq KQ(t,\delta(t))^{\frac{4}{3}}\int_{t-\delta(t)}^{t} I_{\{|X(s;t-\delta(t))-\xi(s)|<2L\}}(s)ds \\ &\leq CQ(t,\delta(t))^{\frac{4}{3}}LQ(t,\delta(t))^{-1} \\ &\leq \frac{CQ(t,\delta(t))^{\frac{4}{3}}}{P(t)^{\frac{2}{3}}Q(t,\delta(t))} \leq \frac{C}{Q(t,\delta(t))^{\frac{1}{3}}} \leq C\delta(t)\frac{P(t)^{\frac{16}{21}}}{Q(t,\delta(t))^{\frac{1}{3}}} \leq C\delta(t)P(t)^{\frac{16}{21}}, \end{aligned} \tag{2.21}$$

since $Q(t,\delta(t)) \geq 1$. Finally, gathering (2.11), (2.12), (2.16), (2.17), (2.20), (2.21) and the definition of $P(t)$ imply the expected lower bound (2.10) . □

3 Proof of Theorem 1.1

Theorem 1.1 now follows from Propositions 2.2 and 2.3 with (2.5), since if $[t_i - t_{i-1}] \leq \delta(t)$, then $\frac{1}{2}n\delta(t) \leq t \leq n\delta(t)$. On the other hand, we have

$$\begin{aligned}\int_0^t (|E(s,X(s;0))| + |E(s,\xi(s))|)ds &\leq \sum_{i=0}^{n-1} \int_{t_i}^{t_i+\delta(t)} (|E(s,X(s;t_i))| + |E(s,\xi(s))|)ds \\ &\leq \sum_{i=0}^{n-1} CP(t)^{\frac{16}{21}}\delta(t) \\ &\leq CP(t)^{\frac{16}{21}} n\delta(t) \leq 2CP(t)^{\frac{16}{21}} t, \text{ for} \delta(t) < t \qquad (3.1)\end{aligned}$$

for any $i = 1, ..., n$, $Q(t_i, \delta(t)) \geq (128K)^2$ and $\delta(t) < t$. Using (2.7) with definition of $P(t)$, Q_0 and (3.1), we obtain $P(t) \leq Q_0 + CP(t)^{\frac{16}{21}} t$, then we can argue exactly as in the case without point to show that

$$P(t) \leq 2CQ_0(1+t)^{\frac{21}{5}} \quad t > 0,$$

since for all $i = 1, ..., n$, $Q(t_i, \delta(t)) \leq (128K)^2$, by the definition of $P(t)$ and $Q(t, \delta(t))$, we have that

$$P(t) \leq \max_{i=1,\ldots,n} Q(t_i, \delta(t)) \leq (128K)^2 \leq C \leq C(1+t)^{\frac{21}{5}}. \qquad (3.2)$$

Finally, let $(X, V)(t) = (X, V)(t_{i-1}, t, x, v)$ be a trajectory such that

$$\sqrt{h}(\bar{t}, X(\bar{t}), V(\bar{t})) = P(\bar{t}) \quad \text{for some } \bar{t} \in [t_{i-1}, t_i].$$

If we choose $\bar{t} = \frac{1}{16k}$, such that $\bar{t} \leq \delta(\bar{t})$, we get $P(\bar{t}) \leq 2^{\frac{16}{21}}$ which implies $P(\bar{t}) \leq 2CQ_0(1+\bar{t})^{\frac{21}{5}}$. Now to extended the result for any $\bar{t} \leq t < \delta(t)$ by definition of $\delta(t)$ and $\bar{t}$, we get $P(t) \leq 2^{\frac{16}{21}}$. As a result of the monotonic increasing of $P(t)$, we have that

$$P(t) \leq CQ_0 + C(1+t)^{\frac{21}{5}} \leq 2CQ_0(1+t)^{\frac{21}{5}} \quad \text{for } 0 < t < \delta(t).$$

The prove of Theorem 1.1 is complete. □

4 Conclusion

Global existence and uniqueness of a classical solution to the three dimensional Vlasov-Poisson system in presence of point charges in case of repulsive interaction is the topic of current interest for many researchers as it has several diversified applications. We show here the size of the velocity support of the distribution function grows at most like $t^{\frac{21}{5}}$, $t \geq 0$.

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Competing Interests

Author has declared that no competing interests exist.

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