British Journal of Mathematics & Computer Science
14(5): 1-7, 2016, Article no.BJMCS.23947
ISSN: 2231-0851

SCIENCEDOMAIN *international*
www.sciencedomain.org

Approximations of the Number π Using Onscribed Regular Polygons

Mieczysław Szyszkowicz[1*]

[1] *112 Four Seasons Drive, Ottawa, K2E 7S1, Canada.*

Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/BJMCS/2016/23947
Editor(s):
(1) Kai-Long Hsiao, Taiwan Shoufu University, Taiwan.
Reviewers:
(1) Radosaw Jedynak, Kazimierz Pulaski University of Technology and Humanities, Poland.
(2) Alexandre Goncalves Pinheiro, Ceara State University, Brazil.
(3) Anonymous, University of Missouri-St. Louis, USA.
Complete Peer review History: http://sciencedomain.org/review-history/13268

Received: 30th December 2015
Accepted: 23rd January 2016
Published: 11th February 2016

Original Research Article

Abstract

Circa 255 B.C., Archimedes invented a method for approximating the value of the number π. He used the perimeters of the inscribed and circumscribed regular polygons to approximate the perimeter of a circle. Starting with two regular hexagons, he doubled the number of their sides up to 96. This approach allowed him to obtain lower and upper estimations of π. He showed that its value lies in the interval $[3 + 10/71, 3 + 1/7]$. Here the use of onscribed regular polygons is proposed for a similar purpose. The onscribed regular polygons are placed between the two polygons used in Archimedes' method. Their location is unique and well defined by applying a criterion to minimize distances. The sequences of areas and perimeters produced by these regular polygons, and their linear combinations, generate values which better approximate π than many other geometrical methods.

Keywords: Archimedes; approximation; circle; polygon; π; quadrature.

2010 Mathematics Subject Classification: 11A03, 11A04, 01A08.

Corresponding author: E-mail: mszyszkowicz@yahoo.ca

1 Introduction

In the ancient world, there were two different approaches for estimating the area of a circle. In the Babylonian (Sumerian) civilization a circle was perceived as a geometrical shape limited by a given perimeter, in a very similar way as a rectangle is limited and determined by its sides. Our modern concept of a circle, obtained as the result of a segment rotation around one of its ends, was probably too abstract for Sumerians. Consequently, their method to calculate the area was based on the perimeter of a circle. They used the following simple rule for this purpose: the area was one twelfth of the squared length of the perimeter (In their numerical system with base 60, the number 5 ($5/60 = 1/12$) was used to multiply $C * C$, where C is the length of the perimeter.) This approach gives $\pi = 3$. In ancient Egypt, the method used to estimate the area was closer to our concept of a circle. Their method to calculate the area used the diameter. The area of the circle was expressed as the area of the square with side of length $8/9d$, where d is the length of diameter. This technique was more accurate than the one one used by the Sumerians. It is interesting that two neighbouring and coexisting civilizations used entirely different methods to obtain the area of a circle. Archimedes (287-212 B.C.) showed that the number π is the same in both situations: it can be used to find both the area and the circumfernce of a circle based on its radius he thus merged the two approaches. Circa 255 B.C., Archimedes, in his treatise "On the Measurements of the Circle", proposed the following method for approximating π. His approach had a recursive form and used the perimeters of the inscribed and circumscribed polygons. Starting with regular 6-sided polygons, he doubled the number of their sides and calculated the corresponding perimeters for 12, 24, 48 and 96 sides. He was able to obtain lower and upper bounds for the number π.

In this paper, the following new term is introduced: an onscribed regular polygon. As the inscribed polygons are inside of the circle, and the circumscribed polygons are outside, thus the onscribed polygons are on top of the circle. Their center is the same as the center of the circle. Their location and size are precisely determined.

In this work is considered the unit circle (radius = 1), whose area is π and half of its circumference has length π also. The main goal of this work is to propose better (geometrical) estimations of π. The examples and illustrations are presented for squares (with $n = 4$ sides). Numerical results are reported for the estimation of π generated using regular polygons with 12 sides ($n = 12$). The innovation of this work includes the definition of a new category of regular polygons (onscibed) for the circle. As Archimedes bounded the circle by two types of regular polygons, here a third type of regular polygons is constructed. As the number of sides in the regular polygons grows, they are closer and closer to the circle. It is proposed to calculate the areas and circumferences for the three types of polygons and combine them using Taylor series to determine the adequate coefficients. The presented methods have faster convergence than known classical methods.

2 Materials and Methods

This work was inspired by the materials presented on the web page of Wroclawski Portal Matematyczny - "Poprawianie Archmedesa - Improving Archimedes" [1]. The author of these web pages defined the distance between two flat geometrical figures based on their areas. He considered the total area of the symmetrical differences of two figures as a measurement of their distance.

Definition 2.1. Distances between two planar figures are measured by the area of the symmetrical set difference between them.

As an example, consider two squares, inscribed in and circumscribed on the unit circle. Their distances, measured according to Definition 2.1, are $1.1415\ldots$ for the inscribed square and $0.8584\ldots$ for the circumscribed square, respectively. Another example is presented in Fig. 1. Here the square was placed on the circle in such a way that its intersections with the circle determine the vertices of a regular octagon. Its side has length x and the distance from the center is z. It is relatively easy to determine the distances between the two shapes (octagon and circle) according to Definition 2.1: this distance is x^2. The properties of the figures allow to solve for x ($= \sqrt{2-\sqrt{2}}$) and z ($= \sqrt{2+\sqrt{2}}/2$) .

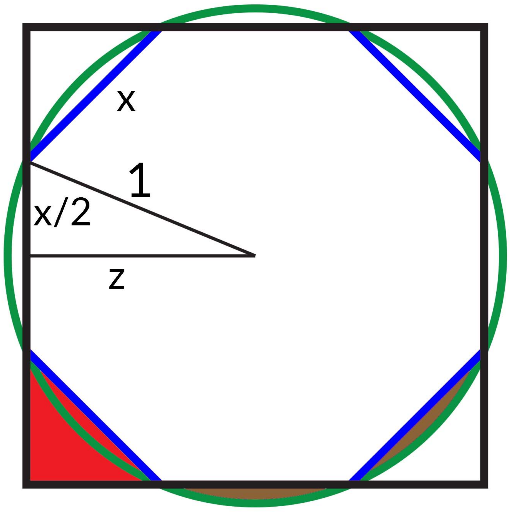

Fig. 1. The square and circle separated by a distance of $2 - \sqrt{2} = 0.5858\ldots$.

Remark 2.1. The distance between a circle and a regular polygon is minimized for the polygon for which its perimeter is divided into two equal parts: one part lying inside, and the other outside of the circle.

Definition 2.2. The regular polygons concentric with a circle and which realize the minimum distance (according to Definition 2.1) are called the onscribed regular polygons.

Fig. 2 shows the onscribed square on the unit circle. To satisfy the definition, the proportion of $2A$ to A (i.e. 2:1) is applied to determine the intersection point of the square and the circle circumference. This intersection point divides the side of the square by two. Both lengths A and B (blue) are equal. The side of an onscribed regular polygon is given by the following trigonometric formula: $S = 2\sin(\pi/n)/\sqrt{3\cos^2(\pi/n)+1}$. From which can be obtained an alternative formula with the sine function only: $S = 2\sin(\pi/n)/\sqrt{4-3\sin^2(\pi/n)}$. Using this formula, we only need to calculate the sine function, which simplifies the numerical calculations. Fig. 3 represents the details for one part (the triangle T) of the considered regular polygons for a given number of sides (n). The corresponding angle is $2\pi/n$, but practically the triangle T is halved into two right-angle triangles. Now the angle $a = \pi/n$ is considered and used to find the sides.

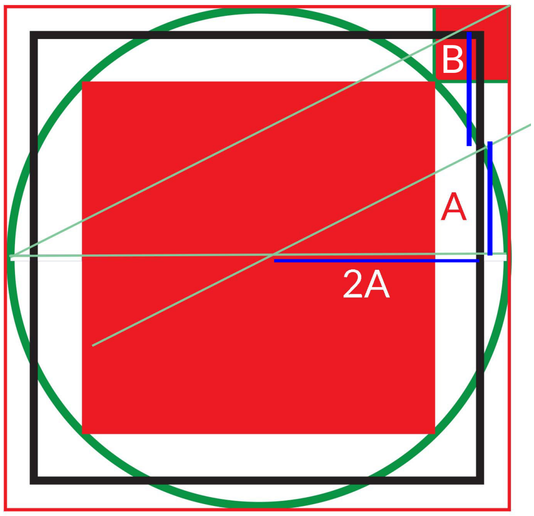

Fig. 2. The circumscribed square (red frame) and inscribed square (red square) for a given circle (green). The onscribed square (black frame). Here $A = \sqrt{5}/5, z = 2A$. The segments are equal: $A = B$

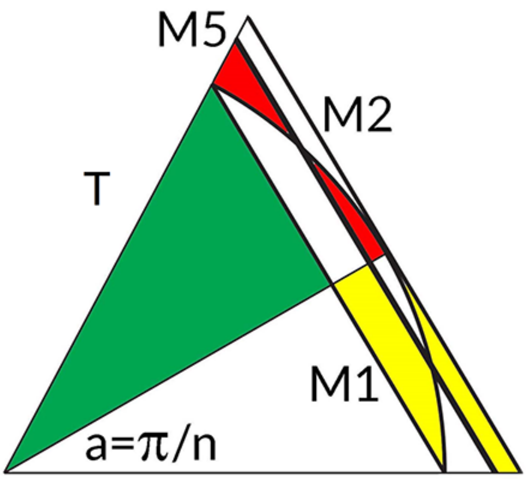

Fig. 3. The sides of the inscribed, circumscribed, and onscribed regular polygons. See Table 1

Based on Fig. 3, it is relatively easy to determine the formula for the area of the triangle which corresponds to the onscribed regular polygon. From trigonometric relations, the area of T is $|T| = 2\sin(2a)/(4 - 3\sin^2(a))$, thus the whole area of the onscribed regular polygon is $n * |T|$.

Table 1. Various methods and their Taylor expansions

M1:	$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362880} - \frac{x^{11}}{39916800} + O(x^{12})$
M2:	$\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \frac{1382x^{11}}{155925} + O(x^{12})$
M3:	$\sin(2x)/2 = x - \frac{2x^3}{3} + \frac{2x^5}{15} - \frac{4x^7}{315} + \frac{2x^9}{2835} - \frac{4x^{11}}{155925} + O(x^{12})$
M4:	$\frac{2\sin(2x)}{4-3\sin^2(x)} = x + \frac{x^3}{12} - \frac{13x^5}{240} - \frac{823x^7}{20160} - \frac{11593x^9}{725760} - \frac{590473x^{11}}{159667200} + O(x^{13})$
M5:	$\frac{2\sin(x)}{\sqrt{4-3\sin^2(x)}} = x + \frac{5x^3}{24} + \frac{61x^5}{1920} - \frac{227x^7}{64512} - \frac{463559x^9}{92897280} - \frac{3597499x^{11}}{1634992128}$
	$- \frac{15352335179x^{13}}{25505877196800} - \frac{300494798603x^{15}}{4284987369062400} + \frac{738301106137201x^{17}}{2331033128769945600 0} + O(x^{13})$

3 Results and Discussion

The following series of mathematical expressions in Table 1 illustrates three classical methods (M1-M3) and formulas for the onscribed regular polygons: M4, their area and M5, the length of their side. In addition, the table shows the Taylor expansion for each method considered. Since $x = \pi/n$, thus the formulas may be used to approximate the value of π by $x*n$. The presented techniques may also be used in geometrical constructions to approximate the unsolvable problem of "the quadrature of the circle". This allows constructing an approximation to the area of the circle.

The results of the performed calculations are listed in Table 2 ($n = 12, x = \pi/n$). The table also presents various linear combinations of the methods, where M1-M5 are the basic ones, and MX1-MX11 are their linear combinations. The Taylor's expansion's terms allow to determine the coefficient for the combined methods (MX) to eliminate lower-power terms in the series. We keep the term with x, as it is used to estimate π. For example the combination $a * M1 + b * M2$ gives two equations: $a + b = 1, -a/6 + b/3 = 0$, and their solution is $a = 2/3, b = 1/3$, which determines the coefficients of method MX2. As a result, this is much better than the arithmetical average (MX1), since it does not contain terms in x^3.

The numerical calculations were done in the R language and a sample program is presented below. The program is not optimal. It is possible to realize this program using only the two functions sine and cosine, since $tan(x) = sin(x)/cos(x)$, and $sin(2x) = 2sin(x)cos(x)$. The best method (MX7) gives the approximation 3.14156... for only $n = 12$ sides. Many other various combinations can be realized to obtain other approximations. A few years ago, one such approach was proposed, using all three methods (M1-M3) to generate more accurate approximations [2]. The author of this work also invented a few new combinations. One of his methods uses the estimations developed by Dörrie [3, 4] and another uses Snell's [5] results for rectification of the arcs [4].

```
The program realizes the following methods: M4, MX5-MX7.
options(digits=15)
for (N in 3:12){
A=pi/N; AN=sin(A)
A2=sin(2*A); AC=cos(A)
BP=N*2*A2/(3*AC*AC+1)
BG=N*tan(A); BI=N*A2*0.5
PG = (4*BP-BG)/3; PI= (8*BP+BI)/9
GI=(7*PI-2*PG)/5
AR=c(BP,PG,PI,GI); print (AR) }
T=c("M4:Area", "MX5:CombTan", "MX6:CombSin","MX7:Comb3")
print(T)
#Results (N=12): 3.15869429739838 3.13979562680668 3.14106159768745 3.14156798603975
```

Probably the first mathematical "publicatio" on the number π was found in Susa (ancient Elam, now Iran). A set of clay tablets from the Old Babylonian Period was excavated at Susa in 1936. One of the clay tablets provides information that gives $\pi = 3\frac{1}{8} = 3.125$. It is interesting that the text on the tablet may be classified as "pure mathematics" in its nature. It does not provide any numerical example, as other clay tablets usually do, but talks in terms of the relation between a circle and a regular hexagon. The mathematical interpretation of the cuneiform text was published in 1950 [6] and latter in 1961 [7]. These published interpretations of how the Babylonians obtained such a value (3.125; a 1 digit precision) are only guesses. Recently, a new idea how the result was obtained was proposed in [8]. The tablet from Susa may be considered as the starting point of geometrical techniques to determine the number π.

Table 2. The methods used and numerical results. ($\pi = 3.14159265358979+$.)

Method (X combined)	Results $\pi = n{*}\mathrm{M}<k>$, $n=12$
M1 - side, inscribed	3.10582854123025
M2 - side, area, circumscribed	3.21539030917347
M3 - area, inscribed	3.00000000000000
MX1=(M1+M2)/2	3.16060942520186
MX2=M1+(M2-M1)/3	3.14234913054466
MX3=(M2+M3)/2	3.10769515458674
MX4=M2+(M3-M2)/3	3.14359353944898
M4 area, onscribed	3.15869429739838
MX5=(4*M4-M2)/3	3.13979562680668
MX6=(8*M4+M3)/9	3.14106159768745
MX7=(7*MX6-2*MX5)/5	3.14156798603975
M5 side, onscribed	3.186916226306593
MX8=(8*M4-5*M2)/3	3.13945942152846
MX9=(4*M4+5*M1)/9	3.14186751237529
MX10=(3*MX8+22*MX9)/25	3.14157854147367
MX11=(16*M1+2*M2-3*M3)/15	3.14160248520206

4 Conclusions

The proposed onscribed regular polygons allow to improve the approximations of the number π. For some situations, their geometrical constructions are also possible.

Acknowledgement

The author would like to thank Sebastian S. Szyszkowicz for his help with editing this paper.

Competing Interests

The author declares that they have no competing interests.

References

[1] Omiljanowski K. Poprawianie Archimedesa. Wroclawski Portal Matematyczny; 2009. Available:http://www.matematyka.wroc.pl/ciekawieomatematyce/poprawianiearchimedesa (Accessed on 19.12.2015)

[2] Chakrabarti G, Hudson R. An improvement of Archimedes method of approximating π. Int. J. Pure Appl. Math., 2003, 7, 207–212.

[3] Dörrie H. 100 Great problems of elementary Mathematics: Their history and solutions. New York: Dover; 1965. (Problem 38).

[4] Szyszkowicz M. Approximations of π and squaring the circle. JMEST. 2015;2(1). Available:http://www.jmest.org/wp-content/uploads/JMESTN42350414.pdf (Accessed on 10.02.2016)

[5] Huygens Ch. De circuli magnitudine inventa, Elzevier, Leiden; 1654.

[6] Bruins EM. Quelques textes mathématiques de la Mission de Suse; 1950, Text: Available:http://www.dwc.knaw.nl/DL/publications/PU00018846.pdf (Accessed on 06.02.2016)

[7] Bruins EM, Rutten M. Textes mathématiques de Suse (Mémoires de la Mission archéologique en Iran, 34). Paris: Paul Geuthner; 1961.

[8] Brette J. Histoire des Mathématiques. Promenade mathématique en Mésopotamie; 2013. Available:http://images.math.cnrs.fr/Promenade-mathematique-en.html (Accessed on 21.01.2016).

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://sciencedomain.org/review-history/13268