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SECURE DOMINATION IN LICT GRAPHS

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Abstract. For any graph $G = (V, E)$, lict graph $\eta(G)$ of a graph G is the graph whose vertex set is the union of the set of edges and the set of cut-vertices of G in which two vertices are adjacent if and only if the corresponding edges are adjacent or the corresponding members of G are incident. A secure lict dominating set of a graph $\eta(G)$, is a dominating set $F \subseteq V(\eta(G))$ with the property that for each $v_1 \in (V(\eta(G)) - F)$, there exists $v_2 \in F$ adjacent to v_1 such that $(F - \{v_2\}) \cup \{v_1\}$ is a dominating set of $\eta(G)$. The secure lict dominating number $\gamma_{se}(\eta(G))$ of G is a minimum cardinality of a secure lict dominating set of G. In this paper, many bounds on $\gamma_{se}(\eta(G))$ are obtained and its exact values for some standard graphs are found in terms of parameters of G. Also its relationship with other domination parameters is investigated.

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1. Introduction

The graphs considered here are finite, connected, undirected without loops or multiple edges and without isolated vertices. As usual n and q denote the number of vertices and edges of a graph G. For any undefined term or notation in this paper can be found in Harary [1].

A set $D \subseteq V$ is a dominating set of G if every vertex in $V - D$ is adjacent to some vertex in D. The dominating number $\gamma(G)$ of G is the minimum cardinality of a dominating set D. A secure dominating set of G is a dominating set $D \subseteq V(G)$

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with the property that for each $u \in V(G) - D$, there exists $v \in D$ adjacent to u such that $(D - \{v\}) - \{u\}$ is a dominating set.
The total domination of lict graph has been studied by [2]. The lict graph $\eta(G)$ of a graph G is the graph whose vertex set is the union of the set of edges and the set of cut-vertices of G in which two vertices are adjacent if and only if the corresponding edges are adjacent or the corresponding members of G are incident. The secure domination has been intensively studied by [3, 4]. The secure lict dominating set of a graph $\eta(G)$, is a dominating set $F \subseteq V(\eta(G))$ with the property that for each $v_1 \in (V(\eta(G)) - F)$, there exists $v_2 \in F$ adjacent to v_1 such that $(F - \{v_2\}) \cup \{v_1\}$ is a dominating set of $\eta(G)$. The secure lict dominating number $\gamma_{se}(\eta(G))$ of G is a minimum cardinality of secure lict dominating set of graph G. For complete review on the topic of domination[5].
The vertex independence number $\beta_0(G)$ is the maximum cardinality among the independent set of vertices of G. $L(G)$ is the line graph of G, $\gamma_e(G)$ is edge domination number, $\gamma_s^{'}(G)$ is the secure edge dominating number, $\gamma_t(G)$ is the total dominating number , $\gamma_{ns}(G)$ is the non-split dominating number and $\chi(G)$is the chromatic number of G. The degree of a edge [6] is the number of lines adjacent to it. The minimum (maximum) degree of an edge in G is denoted by $\delta^{'}(\Delta^{'})$. A subdivision of an edge $e = uv$ of a graph G is the replacement of an edge e by a path (u, v, w) where $w \notin E(G)$. The graph obtained from G by subdividing each edge of G exactly once is called the subdivision graph of G and is denoted by $S(G)$. For any real number X,$\lceil X \rceil$ denotes the smallest integer not less than X and $\lfloor X \rfloor$ denotes the greatest integer not greater than X.
In this paper we established the relationship of this concept with the other domination parameters is investigated.

2. Main results

Theorem 2.1. *First we list out the exact values of $\gamma_{se}(\eta(G))$ for some standard graphs:*

(i) *For any cycle C_n with $n \geq 3$ vertices,*

$$\gamma_{se}(\eta(C_n)) = \begin{cases} 1 & n = 3. \\ \lfloor \frac{n+1}{2} \rfloor & n \ncong 0 (mod\ 7). \\ (\frac{3n}{7}) & n \cong 0 (mod\ 7). \end{cases}$$

(ii) *For any path P_n with $n \geq 4$ vertices,$\gamma_{se}(\eta(P_n)) = n - 2$.*
(iii) *For any star graph $K_{1,n}$ with $n \geq 2$ vertices, $\gamma_{se}(\eta(K_{1,n})) = 1$.*
(iv) *For any wheel graph W_n with $n \geq 4$ vertices, $\gamma_{se}(\eta(W_n)) = \lceil \frac{3n}{7} \rceil + 1$.*
(v) *For any bipartite graph $K_{m,n}$ with $m, n \geq 2$ vertices, $\gamma_{se}(\eta(K_{m,n})) = min\{m, n\}$.*
(vi) *For any friendship graph F_n with k blocks, $\gamma_{se}(\eta(F_n)) = k$.*
(vii) *For any complete graph $K_n, n \geq 4$, $\gamma_{se}(\eta(K_n)) = \lceil \frac{n}{2} \rceil$.*

Theorem 2.2. *Let G be the connected graph with $n \geq 3$ vertices, then $\gamma_{se}(\eta(G)) = 1$ if and only if $G = K_{1,n-1}$ or C_3.*

Proof. Necessary: Suppose $\gamma_{se}(\eta(G)) = 1$. We consider the following cases:

Case 1: If G is a connected graph with $n = 3$, then G is either $K_{1,2}$ or C_3, by using Theorem 2.1(i) and Theorem 2.1(iii), $\gamma_{se}(\eta(G)) = 1$.

Case 2: If G is a connected graph with $n \geq 4$. Let $D = \{e\}$ be the secure dominating set of $\gamma_{se}(\eta(G))$. To prove that $G = K_{1,n-1}$, we assume contrary that $G \neq K_{1,n-1}$. We consider the following subcases:

Subcase 1: Let $F = K_{1,n-1}$ and let the endvertices $v_1, v_2 \in V(F)$, such that the graph G is obtained form F by adding the edge $e_1 = (v_1, v_2) \notin E(F)$. It follows that the set $(D - \{e\}) \cup \{e_1\}$ is not a dominating set of $\eta(G)$. This implies that D is not a dominating set of $\eta(G)$, which is a contradiction. Thus $G = K_{1,n-1}$.

Subcase 2: Let $F = K_{1,n-1}$ and an endvertex $v_1 \in V(F)$, such that the graph G is obtained form F by adding the vertex $v \in V(F)$and the edge $e_1 = (v, v_1)$. It follows that the set $(D-\{e\}) \cup \{e_1\}$ is not an secure dominating set of $\eta(G)$. This implies that D is not a dominating set of $\eta(G)$, which is a contradiction. Thus $G = K_{1,n-1}$.

Sufficiency: If $G = K_{1,n-1}$ or $G = C_3$, then using Theorem 2.1(i)and Theorem 2.1(iii),$\gamma_{se}(\eta(G)) = 1$. □

Theorem 2.3. *For any graph G, $\gamma_{se}(\eta(G)) \geq \gamma_s^{'}(G)$. Equality holds if G is non-separable.*

Proof. Let D be a secure edge dominating set of G and let B be the corresponding vertices of D in $\eta(G)$. We consider the following cases:

Case 1: Suppose the cut-vertices of G are incident with atleast one edge of D in G.
Then for each cut-vertex say v_i in G if there exists an vertex $v \in B, v \in N(v_i)$ in $\eta(G)$ such that $(B - \{v\}) \cup \{v_i\}$ is the dominating set of $\eta(G)$. Then $\gamma_{se}(\eta(G)) = \gamma_s^{'}(G)$. Otherwise $B \cup \{v_i\}$ is the secure dominating set of $\eta(G)$. Therefore $\gamma_{se}(\eta(G)) \geq \gamma_s^{'}(G)$.

Case 2: Suppose if there exists atleast one cut-vertex v_i in G which is not incident with any edge of D, then $B \cup \{v_i\}$ is the secure dominating set of $\eta(G)$. Therefore $\gamma_{se}(\eta(G)) \geq \gamma_s^{'}(G)$.

To prove the equality:
If G is non-separable, then $\eta(G) = L(G)$. Hence $\gamma_{se}(\eta(G)) = \gamma_{se}(L(G)) = \gamma_s^{'}(G)$. □

Theorem 2.4. *For any graph G, $\gamma_{se}(\eta(G)) \geq \gamma_e(G)$.*

Proof. Let D be a γ_e set of graph G. If D is a secure dominating set of graph $\eta(G)$, then

(i) For each $e_i \in E(G) - D$, if there exists an edge $e_1 \in D$, $e_1 \in N(e_i)$, such that the corresponding vertices of $\{(D-e_1) \cup e_i\}$ in $\eta(G)$ is a dominating set of $\eta(G)$.

(ii) For each cut-vertex $c_i \in G$, if there exists an edge $e_i \in D$, c_i is incident with e_i in G such that the corresponding vertices of $\{(D - \{e_i\}) \cup c_i\}$ in $\eta(G)$ is a dominating set of $\eta(G)$.

Therefore $\gamma_{se}(\eta(G)) = \gamma_e(G)$. Otherwise the corresponding vertices of $\{D \cup e_i \cup c_i\}$ in $\eta(G)$ is the secure dominating set of $\eta(G)$. Therefore $\gamma_{se}(\eta(G)) \geq \gamma_e(G)$. □

Theorem 2.5. *For any Tree T, $\gamma_{se}(\eta(T)) \leq m$, where $'m'$ is the number of cut-vertices of T. Equality will holds for P_n and $K_{1,n}$.*

Proof. Let A be the set of cut-vertices of a graph T with $|A| = m$. Since A covers all the edges of T, therefore in $\eta(T)$, A covers all the vertices of $\eta(T)$. Hence the set A is the lict dominating set of T. Now for each vertex $e_i \in V(\eta(T)) - A$, there exists a vertex $\{c_i\} \in A$ incident with $\{e_i\}$ in T such that $(A - \{c_i\}) \cup \{e_i\}$ is a lict dominating set of T. Therefore $\gamma_{se}(\eta(T)) \leq m$.
For Equality:
The result follows from Theorem 2.1(ii) and Theorem 2.1(iii). □

Theorem 2.6. *For any Tree T , $\gamma_{se}(\eta(T)) + 1 \geq \chi(T)$.*

Proof. We have $\chi(T) = 2$ and $\gamma_{se}(T) + 1 \geq 2$, the result follows. □

Theorem 2.7. *If every vertex of G is adjacent to an end vertex then, $\gamma_{se}(\eta(G)) = m$, where m is the number of cut-vertices of G.*

Proof. Let A be the set of cut-vertices of G with $|A| = m$ and let $B = \{e_i/e_i$ is incident with $(v_i, v_j), v_i \in A, d(v_j) = 1\}$ with $|B| = m$. Suppose $\gamma_{se}(\eta(G)) < m$, then A is not the dominating set of $\eta(G)$. Hence $\gamma_{se}(\eta(G)) \geq m$. Now if $\gamma_{se}(\eta(G)) = |A|$, then for each edge $e_i \in E(G) - B$, there exists an edge $e_j \in N(e_i) \cap B$ such that the corresponding vertices of $\{(B - e_j) \cup e_i\}$ is the dominating set of $\eta(G)$ and for every cut-vertex $v_i \in A$, there exists an edge $e_i \in B$ incident with v_i such that the corresponding vertices of $\{(A - e_j) \cup v_i\}$ in $\eta(G)$ is the dominating set of $\eta(G)$. Hence $\gamma_{se}(\eta(G)) = m$. □

Corollary 2.8. *If every vertex of G is adjacent to an end vertex, then $\gamma_{se}(\eta(G)) = \gamma(G) = \gamma_t(G) = \gamma_{ns}(G)$.*

Proof. Since every vertex of G is adjacent to an end vertex then, $\gamma(G) = \gamma_t(G) = \gamma_{ns}(G) = m$ and by using Theorem 2.7, the result follows. □

Theorem 2.9. *For any connected graph G, $\gamma_{se}(\eta(G)) \leq q - \Delta^{'}(G) + 1$, $q \geq 2$ and $\Delta^{'}$ is the maximum degree of an edge.*

Proof. Let e be an edge with degree $\Delta^{'}$ and let S be the set of edges adjacent to e in G. Then $E(G) - S$ is the lict dominating set of G. We consider the following cases:

Case 1: If G is a non-separable graph, then for every edge $e_1 \in S$, there exist an edge $e_2 \in E(G) - S$, $e_1 \in N(e_2)$ such that the corresponding vertices of $\{(E(G) - S) - e_2\} \cup \{e_1\}$ in $\eta(G)$ is a dominating set of $\eta(G)$. Therefore $\gamma_{se}(\eta(G)) \leq q - \Delta^{'}(G)$.

Case 2: If G is a separable graph. We consider the following subcases:

(i) If $((E(G) - S) \cup e)$ is a null graph in G, then $\Delta^{'}(G) = q - 1$ and in the graph G, $e = v_1v_2$, where v_1 or v_2 or both, are the cut-vertices of graph G. Therefore by using Theorem 2.7 , $\gamma_{se}(\eta(G)) = |\{v_1 \cup v_2)\}|$ or $|\{v_1\}|$ or $|\{v_2\}| = 2$ or 1. Therefore $\gamma_{se}(\eta(G)) \leq q - \Delta^{'}(G) + 1$.

(ii) If $((E(G) - S) \cup e)$ is not a null graph and $e_1 = (v_1, v_2)$ where v_1, v_2 are the cut-vertices of the graph G. Now if $v_1 \notin \gamma_{se}$ set of $\eta(G)$, then v_2 is not covered by any vertex corresponding to the vertices of $E(G) - S$ in $\eta(G)$. Therefore the corresponding vertices of $(E(G) - S) \cup \{v_1\}$ in $\eta(G)$ is the secure dominating set of $\eta(G)$. Therefore $\gamma_s(\eta(G)) \leq |(E(G) - S) \cup v_2| = q - \Delta^{'}(G) + 1$. Otherwise if $\{v_1\}$ is the cut-vertex , then there exist an edge $e \in E(G) - S$, e is incident with v_1 such that the corresponding vertices of $\{(E(G) - S) - e) \cup v_1\}$ in $\eta(G)$ is a dominating set of $\eta(G)$. Otherwise if $\{v_1, v_2\}$ are not the cut-vertices then for each vertex e_1 in $\eta(G)$, there exists a vertex $e_2 \in E(G) - S$ in $\eta(G)$, such that the corresponding vertices of $\{(E(G) - S) - e_2) \cup e_1\}$ in $\eta(G)$ is a dominating set of $\eta(G)$. Therefore $\gamma_{se}(\eta(G)) \leq q - \Delta^{'}(G)$.

The result follows from Case (1) and Case (2). □

Theorem 2.10. *For any connected graph* G, $\gamma_{se}(\eta(G)) \leq q - 2, n \geq 3$.

Proof. Let $E(G) = \{e_1, e_2, ...e_m\}$ and let $A = \{e_2, e_3.....e_{m-1}\}$ with$|A| = q-2$. We consider the following cases.

Case 1: If $e_i \in E(G) - A$ is not incident with the cut-vertex say v_i, then for each $e_i \in E(G) - A$ there exists an edge $e_j \in N(e_i) \cap A$ such that the corresponding vertices of $\{(A - e_j) \cup e_i\}$ in $\eta(G)$ is dominating set of $\eta(G)$. Therefore $\gamma_s(\eta(G)) \leq |A| = q - 2$.

Case 2: If $e_i \in E(G) - A$ is incident with the cut-vertex say v_i, then for each v_i, there exists an edge e_j which is incident with v_i and $e_j \in A$ such that the corresponding vertices of $\{(A - e_j) \cup v_i\}$ in $\eta(G)$ is dominating set of $\eta(G)$. Therefore $\gamma_{se}(\eta(G)) \leq |A| = q - 2$.

□

Theorem 2.11. *For any connected graph* G, $\gamma_{se}(\eta(G)) \leq n - 2, n \geq 3$.

Proof. Let $V(G) = \{v_1, v_2, ...v_m\}$,$A = \{v_2, v_3.....v_{m-1}\}$ with $|A| = n - 2$. Let F be the set of edges which is incident with $\{v_{m-1}, v_m\}$ and $B = \{e_i/e_i \in E(G) - F\}$ with $|B| \leq n - 2$. We consider the following cases.

Case 1: If $e_i \in B$ is not incident with the cut-vertex say v_i, then for each $e_i \in E(G) - B$ there exists an edge $e_j \in N(e_i) \cap B$ such that the corresponding vertices of $\{(B - e_j) \cup e_i\}$ is dominating set of $\eta(G)$. Therefore $\gamma_{se}(\eta(G)) \leq |B| = n - 2$.

Case 2: If $e_i \in B$ is incident with the cut-vertex say v_i, then for each v_i, there exists an edge e_j which is incident with v_i and $e_j \in B$ such that the corresponding vertices of $\{(B - e_j) \cup v_i\}$ is dominating set of $\eta(G)$. Therefore $\gamma_{se}(\eta(G)) \leq |B| = n - 2$.

□

Theorem 2.12. *For any connected graph G, $\gamma_e(G) \leq \gamma_{se}(\eta(G)) \leq \gamma_e(G) + \lceil \frac{n}{3} \rceil$.*

Proof. Let D be the edge dominating set of G. Using Theorem 2.4, we have $\gamma_e(G) \leq \gamma_{se}(\eta(G))$. For the upper bound, Let $A = \{v_i/v_i$ is not incident with any edge of $D\}$ with $|A| \leq \lceil \frac{n}{3} \rceil$. Let $B = \{e_i/e_i$ is incident with each $v_i, v_i \in A\}$ with $|B| = |A|$. We consider the following cases:

Case 1: If G is non-separable and for each edge $e_i \in E(G) - D$, there exists an edge $e_j \in N(e_i) \cap D$ such that $\{(D - e_j) \cup e_i\}$ is the edge dominating set of G, then $\gamma_{se}(\eta(G)) = |D| = \gamma_e(G)$. Otherwise if the corresponding vertices of B in $\eta(G)$ does not belong to $\gamma_{se}(\eta(G))$ set, then there exists atleast one vertex in $\eta(G)$ which is not covered by any of the corresponding vertex of D in $\eta(G)$. Therefore $B \in \gamma_{se}(\eta(G))$ set. Hence $\gamma_{se}(G) \leq |D \cup B| = \gamma_s(G) + \lceil \frac{n}{3} \rceil$.

Case 2: If G is separable and if for each edge $e_i \in E(G) - D$ there exists an edge $e_j \in N(e_i) \cap D$ such that the corresponding vertices of $\{(D - e_j) \cup e_i\}$ in $\eta(G)$ is the dominating set of $\eta(G)$ and for each cut-vertex v_i, there exists an edge e_m incident with v_i in G such that the corresponding vertices of $\{(D - e_m) \cup v_i\}$ in $\eta(G)$ is the dominating set of $\eta(G)$. Hence $\gamma_{se}(\eta(G)) = \gamma_e(G)$. Otherwise if the corresponding vertices of B in $\eta(G)$ does not belong to $\gamma_s(\eta(G))$ set, then there exists atleast one vertex in $\eta(G)$ which is not covered by any vertex corresponding to D in $\eta(G)$. Therefore $\gamma_{se}(\eta(G)) \leq |D \cup B| = \gamma_e(G) + \lceil \frac{n}{3} \rceil$.

□

Theorem 2.13. *For any connected graph G, $\gamma_{se}(\eta(G)) \leq \gamma_s'(G) + m$, where $'m'$ is the number of cut-vertices of G.*

Proof. Let D be a secure edge dominating set of G and let $A = \{v_i/G - v_i$ is disconnected $\}$ with $|A| = m$. We consider the following cases:

Case 1: If $m = 0$.
In this case $\eta(G) = L(G)$, the result is obvious.

Case 2: If $m \neq 0$.
Let C be the corresponding vertices of D in $\eta(G)$ and let $F = \{e_i = (v_i, v_j)/e_i \in D$ incident with $v_i \in A\}$. We consider the following subcases:

(i) If $|F| \neq \phi$ and if v_j is a not a cut-vertex or $e_j \in N(e_i) \in D$, then $\gamma_{se}(\eta(G)) = \gamma_s^{'}(G)$. Otherwise there exists atleast one vertex in $\eta(G)$ which is not covered by C. Therefore $A \in \gamma_{se}$ set of $\eta(G)$. Hence $\gamma_{se}(\eta(G)) \leq |C \cup A| = \gamma_s^{'}(G) + m$.

(ii) If $|F| = \phi$, then $v_i \in A$ is not covered by any vertex of C. Therefore $v_i \in \gamma_{se}$ set of $\eta(G)$. Hence $\gamma_{se}(\eta(G)) \leq |C \cup A| = \gamma_s^{'}(G) + m$

□

Theorem 2.14. *For any Tree T, if every vertex is adjacent to support vertex then, $\gamma_{se}(\eta(T)) \leq \beta_0(T)$.*

Proof. Let D be the maximum vertex independence set of T and let A be the γ_{se} set of $\eta(G)$. Since every vertex is adjacent to an end vertex, then A will contains all the vertices adjacent to endvertices of T with $|A| \leq \beta_0(T)$. Therefore by using Theorem 2.7, we have $\gamma_{se}(\eta(T)) \leq \beta_0(T)$. □

Theorem 2.15. *For any connected graph G, $\gamma_{se}(\eta(G)) \leq \alpha_0(G) + m$, where $'m'$ is the number of cut-vertices of G.*

Proof. Let D be the minimum vertex covering set of G and B be the set of cut-vertices of G with $|B| = m$. For each vertex $v_i \in D$, choose exactly one edge $e_i \in G, e_i$ is incident with v_i such that it covers maximum number of vertices of G. Let F be the set of all such edges such that $|F| = |D|$. We consider the following cases:

Case 1: If $m = 0$.
For each edge $e_i \in E(G) - F$, there is an edge $e_j \in F, e_j \in N(e_i)$. Since the corresponding vertices of F in $\eta(G)$ will covers all the edges which are incident to $v_i \in V(G)$. Therefore the corresponding vertices of $\{(F - e_j) \cup e_i\}$ in $\eta(G)$ is the dominating set of $\eta(G)$. Hence $\gamma_{se}(\eta(G)) \leq \alpha_0(G)$.

Case 2: If $m \neq 0$.
Let S the corresponding vertices of F in $\eta(G)$. If $v_i \in B \cap D$, then for each vertex v_i, there exists an edge $e_j, e_j \in F$ and e_j is incident with $\{v_i, v_j\}, v_j$ is not a cut-vertex, then corresponding vertices of $\{(F - e_j) \cup v_i\}$ in $\eta(G)$ is the dominating set of $\eta(G)$. Therefore $\gamma_{se}(\eta(G)) \leq \alpha_0(G)$. Otherwise there exists atleast one vertex $v_i \in B \cap V(\eta(G))$ which is not covered by S, therefore $v_i \in \gamma_{se}(\eta(G)$ set of $\eta(G)$. Therefore $\gamma_{se}(\eta(G)) \leq |D \cup B| = \alpha_0(G) + m$.

The result follows from Case (1) and Case (2). □

Theorem 2.16. *For any connected graph G, $\gamma_{se}(\eta(G)) \leq \alpha_1(G) + m$, where m is the number of cut-vertices of G.*

Proof. Let D be the minimum edge covering set of G and B be the set of cut-vertices of G with $|B| = m$. We consider the following cases:

Case 1: If $e_i \in D$ is incident with $\{v_1, v_2\}$, $(v_1, v_2) \notin B$.
For each edge $e_j \in E(G)-D$, there exists an edge $e_i \in D, e_i \in N(e_j)$ and since e_j covers all the edges incident with v_1 and $(D - e_i)$ will covers all the edges incident with v_2, therefore the corresponding vertices of $\{(D-e_i)\cup e_j\}$ in $\eta(G)$ is the dominating set of $\eta(G)$. Hence $\gamma_{se}(\eta(G)) \leq \alpha_1(G)$.

Case 2: If $e_i \in D$ is incident with $\{v_1, v_2\}$, $(v_1, v_2) \in B$
For each edge e_i, if there exists an edge adjacent to e_i in $E(G) \cap D$, then $\gamma_s(\eta(G)) \leq \alpha_1(G)$. Otherwise $\{v_1\}$ or $\{v_2\}$ is not covered by any vertices corresponding to D in $\eta(G)$. Therefore $\{v_1 \text{ or } v_2\} \in \gamma_s(\eta(G))$ set. Hence $\gamma_{se}(\eta(G)) \leq |D \cup B| = \alpha_1(G) + m$.

Case 3: If $e_i \in D$ is incident with $\{v_1, v_2\}$, $\{v_1\} \in B$,$\{v_2\} \notin B$
For each edge $e_j \in E(G) - D$, there exists an edge $e_i, e_i \in N(e_i)$ and since e_j covers all the edges incident with v_1 and $(D - e_i)$ will covers all the edges incident with v_2, therefore the corresponding vertices of $\{(D-e_i)\cup e_j\}$ in $\eta(G)$ is the dominating set of $\eta(G)$ and for v_1 there exists an edge $e_r \in D$ incident with v_1 such that the corresponding vertices of $\{(D-v_1)\cup e_r\}$ in $\eta(G)$ is the dominating set. Hence $\gamma_{se}(\eta(G)) \leq \alpha_1(G)$.

The result follows from Case (1), Case (2) and Case (3). □

Theorem 2.17. *For any connected graph G, $\gamma_{se}(\eta(G)) \leq \beta_1(G) + m$, where m is the number of cut-vertices of the graph G.*

Proof. Let D be the maximum edge independence set of G and B be the set of cut-vertices of G with $|B| = m$. We consider the following cases:

Case 1: If $e_i \in D$ is incident with $\{v_1, v_2\}$, $(v_1, v_2) \notin B$.
For each edge $e_i \in E(G)-D$, there exists an edge $e_j \in D, e_i \in N(e_i)$ and since e_j covers all the edges incident with v_1 and $(D - e_i)$ will covers all the edges incident with v_2, therefore the corresponding vertices of $\{(D-e_i)\cup e_j\}$ in $\eta(G)$ is the dominating set of $\eta(G)$. Hence $\gamma_{se}(\eta(G)) \leq \beta_1(G)$.

Case 2: If $e_i \in D$ is incident with $\{v_1, v_2\}$, $(v_1, v_2) \in B$
For each edge e_i, if there exists an edge adjacent to e_i in $E(G) \cap D$, then $\gamma_s(\eta(G)) \leq \beta_1(G)$. Otherwise $\{v_1\}$ or $\{v_2\}$ is not covered by any vertices corresponding to D in $\eta(G)$. Therefore $\{v_1 \text{ or } v_2\} \in \gamma_s(\eta(G))$ set. Hence $\gamma_{se}(\eta(G)) \leq |D \cup B| = \beta_1(G) + m$.

Case 3: If $e_i \in D$ is incident with $\{v_1, v_2\}$, $\{v_1\} \in B$,$\{v_2\} \notin B$
For each edge $e_i \in E(G)-D$, there exists an edge $e_j \in D, e_i \in N(e_i)$ and since e_j covers all the edges incident with v_1 and $(D - e_i)$ will covers all the edges incident with v_2, therefore the corresponding vertices of $\{(D - e_i) \cup e_j\}$ in $\eta(G)$ is the dominating set of $\eta(G)$ and for v_1 there exists an edge $e_r \in D$ incident with v_1 such that the corresponding vertices of $\{(D - v_1) \cup e_r\}$ in $\eta(G)$ is the dominating set of G. Hence $\gamma_{se}(\eta(G)) \leq \beta_1(G)$.

The result follows from Case (1), Case (2) and Case (3). □

Theorem 2.18. *For any graph G, $\gamma(G) \leq \gamma_{se}(\eta(G)$.*

Proof. Let D be the γ set of G and let $B = \{e_i/e_i \in E(G)$ incident with D such that it covers maximum number of vertices and $|D| = |B|$. If suppose B is the secure dominating set of G, then

(i) For each $e_i \in E(G) - D$, if there exists an edge $e_1 \in D$, $e_1 \in N(e_i)$, such that the corresponding vertices of $\{(D - e_1) \cup e_i\}$ in $\eta(G)$ is a dominating set of $\eta(G)$.

(ii) For each cut-vertex $c_i \in G$, if there exists an edge $e_i \in D$, c_i is incident with e_i in G such that the corresponding vertices of $\{(D - \{e_i\}) \cup c_i\}$ in $\eta(G)$ is a dominating set of $\eta(G)$.

Therefore $\gamma_{se}(\eta(G)) = \gamma(G)$. Otherwise the corresponding vertices of $\{D \cup e_i \cup c_i\}$ in $\eta(G)$ is the secure dominating set of $\eta(G)$. Therefore $\gamma(G) \leq \gamma_{se}(\eta(G))$. □

Theorem 2.19. *For any connected graph G, $\frac{n}{\Delta+1} \leq \gamma_{se}(\eta(G)) \leq 2q - n$. Furthermore, the upper bound is attained if and only if G is a path.*

Proof. Since $\frac{n}{\Delta+1} \leq \gamma(G)$ and using Theorem 2.18. $\gamma(G) \leq \gamma_{se}(\eta(G))$, the lower bound holds.
For upper bound.
Since G is connected, $q \geq n - 1$ and by Theorem 2.11, $\gamma_{se}(\eta(G)) \leq n - 2 = 2(n-1) - n$. Hence $\gamma_{se}(\eta(G)) \leq 2q - n$.
Now we show that $\gamma_{se}(\eta(G)) = 2q - n$ if and only if G is path. If G is a path, then by Theorem 2.1(iii), $\gamma_{se}(\eta(G)) = n - 2 = 2(n-1) - n = 2q - n$. conversely, suppose $\gamma_{se}(\eta(G)) = 2q - n$. Then by Theorem 2.11, we have $2q - n \leq n - 2$ which implies $q \leq n - 1$. Since G is connected, G must be a tree with $q = n - 1$. Thus Theorem 2.5, $\gamma_{se}(\eta(G)) \leq n - e, e$ is the number of pendent vertices. If $e > 2$, then $\gamma_{se}(\eta(G)) \leq n - e < n - 2 = 2q - n$, a contradiction which shows that $e \leq 2$. But G is a tree, $e \geq 2$. Thus $e = 2$ and G is a path. □

Theorem 2.20. *For any subdivision graph G, $\gamma_{se}(\eta(S(G)) \leq 2\alpha_1 + n_0$, where n_0 is the number of vertices that subdivides $E(G)$.*

Proof. Let $V(G) = \{v_1, v_2, v_3, ..., v_n\}$ and let $B = \{n_j/n_j \in V(S(G)) - V(G)\}$ with $|B| = n_0$. Let α_1 be the minimum edge covering set of G and $M = \{n_r/n_r \in B, n_r$ in not incident with α_1 and n_r is the cut-vertex in $S(G)\}$ with $|M| \leq n_0$. Let R be the set of edges in $S(G)$ corresponding to the edges in α_1 with $|R| = 2\alpha_1$. We consider the following cases:

Case 1: If $|M| = \phi$.
Then for each edge $e_j = (v_i, n_j) \in E(S(G)) - R$, there exists an edge $e_i = (v_i, n_j) \in R \cap N(e_j)$ such that the corresponding vertices of $\{(R - e_i) \cup e_j\}$ in $\eta(S(G))$ is the dominating set of $\eta(S(G))$ and for each cut-vertex $v_i \in V(S(G))$, there exists an edge $e_j = (v_i, n_j) \in R$,e_j is incident with v_i such that the corresponding vertices of $\{(R - e_j) \cup v_i\}$ in $\eta(S(G))$ is the dominating set of $\eta(S(G))$. Hence $\gamma_{se}(\eta(S(G))) \leq 2\alpha_1$.

Case 1: If $|M| \neq \phi$.
Now for each cut-vertex say $v_m \in M$, since there is no edge in R incident with v_m and therefore v_m is not covered by the corresponding vertices of R in $\eta(S(G))$. Therefore $v_m \in \gamma_{se}(\eta(S(G))$ set. Hence $\gamma_{se}(\eta(S(G))) \leq 2\alpha_1 + n_0$.

□

Proposition 2.21. *For any graph $G = K_{1,n}$, $\gamma_e(S(G)) = n - 1$.*

Theorem 2.22. *For any graph $G = K_{1,n}$, $\gamma_{se}(\eta(S(G))) = n - 1$.*

Proof. Let $V(G) = \{v_1, v_2, v_3, ..., v_n, d(v_n) = n-1\}$ and $E(G) = \{e_i = (v_n, v_i), i = 1, 2, 3, ..., n-1\}$. Let $B = \{w_i/w_i$ is the vertex subdividing $e_i\}$ and $D = \{(v_n, w_i), i = 1, 2, 3, ..., n-1\}$. The corresponding vertices of D in $\eta(S(G))$ covers all the vertices of $\eta(S(G))$ with $|D| = n-1$. Now for each edge $(w_i, v_i), i = 1$ to $n-1 \in E(S(G)) - D$, there exists an edge $(w_i, v_n) \in D \cap N(w_i, v_i)$, such that the corresponding vertices of $\{(D-(w_i, v_n)) \cup (w_i, v_i))\}$ in $\eta(S(G))$ is the dominating set of $\eta(S(G))$ and for $v_n \in V(S(G))$, there exists an edge $(v_n, w_i), i$ to $n-1$, incident with v_n such that the corresponding vertices of $\{(D - (v_n, w_i)) \cup v_n\}$ in $\eta(G)$ is the dominating set of $\eta(G)$. Hence $\gamma_s(\eta(S(G))) = |D| = n - 1$. □

Proposition 2.23. *For any graph $G = K_n$, $\gamma_e(S(G)) = n - 1$.*

Theorem 2.24. *For any graph $G = K_n$, $\gamma_{se}(\eta(S(G))) = n$.*

Proof. Let $V(G) = \{v_1, v_2, v_3, ..., v_n\}$ and $A = \{n_j/n_j \in V(S(G)) - V(G), j = 1, 2, 3, 4, 5, ..., \}$ with $|A| = \frac{n(n-1)}{2}$. The set $B = \{e_i = (v_i, n_j), i = 1, 2, ..., n-1, j = 1, 2, 3, 4, 5, ..., \frac{n(n-1)}{2}\}$ such that $n_j \in N(v_n)$. The corresponding vertices of B in $\eta(S(G))$ will covers all the vertices of $\eta(S(G))$ with $|B| = n-1$. Suppose if the corresponding vertices of $B \notin \gamma_{se}(\eta(S(G))$ set, then there exists atleast one edge (v_i, n_j) or (v_n, v_j) in $S(G)$ which is not covered by the corresponding vertices of B in $\eta(S(G))$. Now in $S(G)$, consider the set $C = \{B \cup (v_n, n_j)\}$, then for every edge $e_j = (v_i, n_j) \in E(S(G)) - C$ there exists an edge $e_k = (v_i, n_j) \in C \cap N(e_j)$ such that the corresponding vertices of $\{(C - e_k) \cup e_j\}$ in $\eta(S(G))$ is the dominating set of $\eta(S(G))$. Hence $\gamma_{se}(\eta(S(G))) = n$ □

Proposition 2.25. *For any graph $G = K_{m,n}$, $\gamma_e(S(G)) = m + n - 1, m \geq n, m, n \geq 2$.*

Theorem 2.26. *For any graph $G = K_{m,n}$, $\gamma_{se}(\eta(S(G))) = m + n, m \geq n, m \geq 2$.*

Proof. Let $V(G) = \{v_1, v_2, v_3, ..., v_n\}$ and $A = \{n_j/n_j \in V(S(G)) - V(G), j = 1, 2, 3, 4, ..., \}$ with $|A| = mn$. The set $B = \{(v_i, n_j), d(v_i) = m, n_j \in N(v_1), d(v_1) = n\}$ and $C = \{(v_i, n_j), d(v_i) = n, v_i \neq v_1, n_j \in N(v_2), d(v_2) = m\}$. Then the corresponding vertices of $D = B \cup C$ covers all the vertices of $S(\eta(S(G)))$ with $|D| = m + n - 1$. For $e_1 = (v_2, n_j) \in E(S(G) - D$, there exists an edge $e_2 = (v_1, n_j)$ adjacent to e_1 such that $\{(B - e_2 \cup e_1\}$ is a not an edge dominating set of $S(G)$. Now in $S(G)$, the set $E = D \cup (v_i, n_j), d(v_i) = m, v_i \neq v_n$, then for every edge $e_j = (v_i, n_j \in E(S(G)) - D$ there exists an edge $e_k = (v_i, n_j) \in D$ such that $\{(E - e_k) \cup e_j\}$ is the secure lict dominating set of G. Hence $\gamma_{se}(S(G)) = m + n$ □

Proposition 2.27. *For any graph $G = W_n$, $\gamma_e(S(G)) = n - 1$.*

Theorem 2.28. *For any graph $G = W_n$, $\gamma_{se}(\eta(S(G))) = n$.*

Proof. Let $V(G) = \{v_1, v_2, v_3, s..., v_n, d(v_n) = n-1\}$ and $A = \{n_j/n_j \in V(S(W_n)) - V(W_n), j = 1, 2, 3, 4, 5, ..., 2(n-1)\}$ with $|A| = 2(n-1)$. The set $B = \{e_i = (v_i, n_j), i = 1, 2, ..., n-1, j = 1, 2, 3, 4, 5....2(n-1)\}$ such that $n_j \in N(v_n)$. The corresponding vertices of B will covers all the vertices of $\eta(S(G))$ with $|B| = n-1$ and therefore B is the lict dominating set of $S(G)$. If suppose B is the secure lict dominating set of $S(G)$, then there exists atleast one vertex (v_n, n_j) or $(v_i, n_j), i = 1, 2, ..., n-1$ in $\eta(G)$ which is not covered by corresponding vertex of B. Now in $S(G)$, consider the set $C = \{B \cup (v_n, n_j)\}$, then for every edge $e_j = (v_i, n_j) \in E(S(G) - C$ there exists an edge $e_k = (v_i, n_j) \in C \cap N(e_j)$ such that the corresponding $\{(C - e_k) \cup e_j\}$ is the lict dominating set of $S(G)$. Hence $\gamma_{se}(S(G)) = n$ □

Theorem 2.29. *[3]:For any graph G of order $n \geq 3$,*

(i) $\beta_1(G) + \beta_1(\bar{G}) \leqslant 2\lceil \frac{n}{2} \rceil$.
(ii) $\beta_1(G) * \beta_1(\bar{G}) \leqslant \lceil \frac{n}{2} \rceil^2$.

Theorem 2.30. *For any non-separable connected graph G of order $n \geq 3$ vertices,*

(i) $\gamma_{se}(\eta(G)) + \gamma_{se}(\eta(\bar{G})) \leq 2\lceil \frac{n}{2} \rceil$.
(ii) $\gamma_{se}(\eta(G)) * \gamma_{se}(\eta(\bar{G})) \leq \lceil \frac{n}{2} \rceil^2$.

Proof. Since $\beta_1(G) \leq \lceil \frac{n}{2} \rceil$, the result follows from 2.17 and using Theorem 2.29. □

Competing interests

The authors declare that they have no competing interests.

References

1. Harary, F. (1969). Graph theory Addison-Wesley Reading MA USA.
2. Girish, V. R., & Usha, P. (2014). Total Domination in Lict Graph. *International journal of Mathematical combinatorics,* 1, 19-27.
3. Kulli, V.R.(2016). Secure Edge Domination in Graphs. *Journal of Pure and Applied Mathematics, 12*(1), 95-99.
4. Merouane,H.B., & Chellai, M. (2015). On Secure Domination in Graphs. *Information Processing Letters,115* (10),786-790.https://doi.org/10.1016/j.ipl.2015.05.006
5. Teresa, W. H., Stephen, T.H., & Peter, J.S. (1998). Fundamental of domination of graphs, Marcel Dekker, Inc.Newyork.

6. Jayaram, S.R. (1997). Line domination in graphs. *Graph Combin.*, 357-363.

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