*Open J. Math. Sci., Vol. 2(2018),No.1, pp. 93 - 114*
Website: https://pisrt.org/psr-press/journals/oms/
ISSN: 2523-0212 (Online) 2616-4906 (Print)
http://dx.doi.org/10.30538/oms2018.0020

# BOUNDEDNESS OF FRACTIONAL MARCINKIEWICZ INTEGRAL WITH VARIABLE KERNEL ON VARIABLE MORREY-HERZ SPACES

AFIF ABDALMONEM[1], OMER ABDALRHMAN, SHUANGPING TAO

Abstract. In this article, the authors obtain the boundedness of the fractional Marcinkiewicz integral with variable kernel on Morrey-Herz spaces with variable exponents $\alpha$ and $p$. The corresponding boundedness for commutators generalized by the Lipschitz function is also considered.

Mathematics Subject Classification: 42B20, 42B25.
*Key words and phrases:* Fractional Marcinkiewicz integral operators; variable kernel; commutators; Morrey-Herz spaces with variable exponents.

## 1. Introduction

In 1938 Marcinkiewicz [1] introduced the Marcinkiewicz integral. Later, Stain [2] studied the bounded of Marcinkiewicz integral on $L^p(\mathbb{R}^n)$ for any $p \in (0, 1]$ and bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$. Torchinsky and Wang in [3] discussed the boundedness for the commutator generated by the Marcinkiewicz integral $\mu_\Omega$ and $BMO(\mathbb{R}^n)$ function on Lebesgue spaces $L^p(\mathbb{R}^n)$. Pan and Wang [4] established the boundedness of fractional Marcinkiewicz integrals with variable kernels on Hardy type spaces.

Moving in another direction, Kováčik and Rákosník [5] introduced the class of variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$, after that many authors has been interested in studying the function spaces with variable exponents [6, 7, 8, 9, 10, 11]. Izuki [12] and [13] introduced the class of variable exponent Herz-Morrey space $M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ and the boundedness of the sublinear operator satisfying

Received 12 March 2018. Revised 03 May 2018.
[1] Corresponding Author

size condition and fractional integral on those spaces were proved. Hongbin Wang [14] studied the commutator of Marcinkiewicz integrals on Herz spaces with variable exponent. In recent years, Yan Lu and Yue Ping [15] introduced the class of $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ and also established the boundedness of multilinear Calderón-Zygmund singular operators on those spaces.
In this paper, we consider the boundedness of the fractional Marcinkiewicz integral with variable kernel $[\mu_{\Omega,\gamma}]$ and its commutators $[b^m,\mu_{\Omega,\gamma}]$ on $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$. Now assume that $S^{n-1}(n\geq 2)$ the unit sphere in $\mathbb{R}^n$ with the normalized Lebesgue measure $d\sigma(x')$. Let $\Omega\in L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})(r\geq 1)$ be homogeneous of degree zero and satisfy the vanishing condition on $S^{n-1}$, that is

$$\int_{s^{n-1}}\Omega(x,z')d\sigma(z')=0,\ \forall x\in\mathbb{R}^n. \tag{1}$$

The fractional Marcinkiewicz integral operator with variable kernel $\mu_{\Omega,\gamma}(0<\gamma<n)$ is defined by

$$\mu_{\Omega,\gamma}(f)(x)=\left(\int_0^\infty|F_{\Omega,t,\gamma}(x)|^2\frac{dt}{t^3}\right)^{\frac{1}{2}}, \tag{2}$$

where

$$F_{\Omega,t,\gamma}(x)=\int_{|x-y|\leq t}\frac{\Omega(x,x-y)}{|x-y|^{n-\gamma-1}}f(y)dy. \tag{3}$$

Obviously, if $\gamma=0$, $\mu_{\Omega,\gamma}$ is just Marcinkiewicz integral with variable kernels is defined by

$$\mu_{\Omega}(f)(x)=\left(\int_0^\infty\left|\int_{|x-y|\leq t}\frac{\Omega(x,x-y)}{|x-y|^{n-1}}f(y)dy\right|^2\frac{dt}{t^3}\right)^{\frac{1}{2}}. \tag{4}$$

If $\Omega\equiv 1$, then $\mu_{\Omega,\gamma}=\mu_\gamma$ is a fractional Marcinkiewicz integral operator is defined by

$$\mu_{\gamma}(f)(x)=\left(\int_0^\infty\left|\int_{|x-y|\leq t}\frac{f(y)}{|x-y|^{n-\gamma-1}}dy\right|^2\frac{dt}{t^3}\right)^{\frac{1}{2}}. \tag{5}$$

The commutators of the fractional Marcinkiewicz integral operator with variable kernels is defined by

$$[b^m,\mu_{\Omega,\gamma}](f)(x)=\left(\int_0^\infty\left|\int_{|x-y|\leq t}\frac{\Omega(x,x-y)}{|x-y|^{n-\gamma-1}}[b(x)-b(y)]^m f(y)dy\right|^2\frac{dt}{t^3}\right)^{\frac{1}{2}}. \tag{6}$$

Now, we need the further assumption for $\Omega(x,z)$, the integral modulus $\omega_r(\delta)$ of $L^r$ continuity of $\Omega$ is defined by

$$\omega_r(\delta) = \sup_{\|\mu\|<\delta} \left( \int_{s^{n-1}} |\Omega(x,\mu z') - \Omega(x,z')|^r d\sigma(z') \right)^{\frac{1}{r}}, 1 \leq r < \infty$$

and

$$\omega_\infty(\delta) = \sup_{\|\mu\|<\delta} |\Omega(x,\mu z') - \Omega(x,z')|,$$

where $0 < \delta \leq 1$, $\mu$ is the rotation in $s^{n-1}$, $\|\mu\| = \sup_{z' \in S^{n-1}} |\mu z' - z'|$.

Let $E$ be a measurable set in $\mathbb{R}^n$ with $|E| > 0$. Next, we introduce the Lebesgue spaces with variable exponents.

**Definition 1.1.** [5] Let $p(\cdot) : E \to [1,\infty)$ be a measurable function. The Lebesgue space with variable exponent $L^{p(\cdot)}(E)$ is defined by

$$L^{p(\cdot)}(E) = \{f \text{ is measurable} : \int_E \left(\frac{|f(x)|}{\eta}\right)^{p(x)} dx < \infty \text{ for some constant } \eta > 0\}.$$

The space $L^{p(\cdot)}_{loc}(E)$ is defined by

$$L^{p(\cdot)}_{loc}(E) = \{f \text{ is measurable} : f \in L^{p(\cdot)}(K) \text{ for all compact } K \subset E\}.$$

The Lebesgue spaces $L^{p(\cdot)}(E)$ is a Banach spaces with the norm defined by

$$\|f\|_{L^{p(\cdot)}(E)} = \inf\{\eta > 0 : \int_E \left(\frac{|f(x)|}{\eta}\right)^{p(x)} dx \leq 1\}.$$

We denote $p_- = \text{essinf}\,\{p(x) : x \in E\}$, $p_+ = \text{esssup}\{p(x) : x \in E\}$. Then $\mathcal{P}(E)$ consists of all $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$, and

$$\mathfrak{B}(E) = \left\{p(\cdot) \in \mathcal{P}(E) : \text{ M is bounded on } L^{p(\cdot)}\right\},$$

where the Hardy-Littlewood maximal operator $M$ is defined by

$$Mf(x) = \sup_{r>0} r^{-n} \int_{B(x,r)\cap E} |f(y)|dy.$$

We see that a function $\alpha(\cdot) : \mathbb{R}^n \longrightarrow \mathbb{R}$ is called log-Hölder continuous at the origin(or has a log decay at the origin ), if there exists a constant $C_{log} > 0$ such that

$$|\alpha(x) - \alpha(0)| \leq \frac{C_{log}}{log(e + \frac{1}{|x|})},\ \forall x \in \mathbb{R}^n.$$

If for some $\alpha_\infty \in \mathbb{R}$ and $C_{log} > 0$, we have

$$|\alpha(x) - \alpha_\infty| \leq \frac{C_{log}}{log(e + |x|)},\ \forall x \in \mathbb{R}^n.$$

Then $\alpha(\cdot)$ is called log-Hölder continuous at infinity (or has a log decay at infinity).
By $\mathcal{P}_0^{log}(\mathbb{R}^n)$ and $\mathcal{P}_\infty^{log}(\mathbb{R}^n)$ we denote the class of all exponents $p \in \mathcal{P}(\mathbb{R}^n)$ which have a log decay at the origin and at infinity, respectively. It is worth noting that if $p(\cdot) \in \mathcal{P}_0^{log}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{log}(\mathbb{R}^n)$, then we have $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$.
Let $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}, C_k = B_k \backslash B_{k-1}, \chi_k = \chi_{B_k}, k \in \mathbb{Z}$. Almeida and Direhem in [16] introduced the following definition.

**Definition 1.2.** Let $0 < q \leq \infty, p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\alpha(\cdot) : \mathbb{R}^n \longrightarrow \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$.

(1) The homogeneous Herz space $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)$ is defined as the set of all $f \in L_{loc}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\})$ such that

$$\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)} := \left( \sum_{k \in \mathbb{Z}} \|2^{k\alpha(\cdot)} f \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right)^{1/q} < \infty.$$

(2) The nonhomogeneous Herz space $K_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)$ consists of all $f \in L_{loc}^{p(\cdot)}(\mathbb{R}^n)$ such that

$$\|f\|_{K_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)} := \|f\chi_{B_0}\|_{L^{p(\cdot)}(\mathbb{R}^n)} + \left( \sum_{k \geq 1} \|2^{k\alpha(\cdot)} f \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right)^{1/q} < \infty,$$

with the usual modification when $q = \infty$.

**Definition 1.3.** Let $0 < q \leq \infty, p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$,$0 \leq \lambda < \infty$ and $\alpha(\cdot) : \mathbb{R}^n \longrightarrow \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$. The homogeneous Morrey-Herz space $M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)$ is defined as the set of all $f \in L_{loc}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\})$ such that

$$\|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)} := \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left( \sum_{k=-\infty}^{L} \|2^{k\alpha(\cdot)} f \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right)^{1/q} < \infty,$$

with the usual modification when $q = \infty$.

**Remark 1.1.** If $\alpha(\cdot)$ is constant, then $M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n) = M\dot{K}_{q,p(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)$. If $\lambda = 0$, then $M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n) = \dot{K}_{q,p(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$. If both $\alpha(\cdot)$ and $p(\cdot)$ are constant and $\lambda = 0$, then $M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n) = \dot{K}_{q,p}^{\alpha}(\mathbb{R}^n)$ is the classical Herz space introduced in [17].

**Definition 1.4.** [18] For $0 < \beta \leq 1$, the Lipschitz space $Lip_\beta(\mathbb{R}^n)$ is defined by

$$Lip_\beta(\mathbb{R}^n) = \left\{ f : \|f\|_{Lip_\beta} = \sup_{x,y \in \mathbb{R}^n, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\beta} < \infty \right\}.$$

## 2. Properties of variable exponents

Before stating our main results, we introduce some key Lemmas which will be used later. The next Lemma is the generalization of Herz with variable exponents spaces in [16], and it was used by Lu and Zhu in [15].

**Lemma 2.1.** *Let $0 < q \leq \infty$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$,$0 \leq \lambda < \infty$ and $\alpha \in L^\infty(\mathbb{R}^n)$. If $\alpha(\cdot)$ is log-Hölder continuous both at the origin and at infinity, then*

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} \approx max\left\{ \sup_{k_0<0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q}\left\|f\chi_k\right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/q}, \right.$$
$$\sup_{k_0\geq 0,k_0\in\mathbb{Z}}\left[2^{-k_0\lambda}\left(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q}\left\|f\chi_k\right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/q}\right]$$
$$\left. +\left[2^{-k_0\lambda}\left(\sum_{k=0}^{k_0} 2^{k\alpha_\infty q}\left\|f\chi_k\right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/q}\right]\right\}.$$

**Lemma 2.2.** [10]

(1) *If $p \in \mathcal{P}(\mathbb{R}^n)$, then for all $f \in L^{q(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, we have*

$$\int\limits_{\mathbb{R}^n} |f(x)g(x)|dx \leq C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.$$

(2) *If $p(\cdot), q(\cdot), r(\cdot) \in \mathbb{R}^n$, define $p(\cdot)$ by : $\frac{1}{p(\cdot)} = \frac{1}{q(\cdot)} + \frac{1}{r(\cdot)}$. Then there exists a constant $C$ such that for all $f \in L^{q(\cdot)}(\mathbb{R}^n)$, $g \in L^{r(\cdot)}(\mathbb{R}^n)$, we have*

$$\|fg\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{r(\cdot)}(\mathbb{R}^n)}.$$

**Lemma 2.3.** [19] *Suppose that $p_1(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$, $\Omega \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$. Let $0 < \gamma \leq \frac{n}{(p_1)_+}$, and define $p_2(\cdot)$ by: $\frac{1}{p_1(x)} - \frac{1}{p_2(x)} = \frac{\gamma}{n}$, then for all $f \in L^{p_1(\cdot)}(\mathbb{R}^n)$, we have*

$$\|T_{\Omega,\gamma}f\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}.$$

**Lemma 2.4.** [20] *Suppose that $p_1(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$, $b \in Lip_\beta(\mathbb{R}^n)$, $0 < \beta \leq 1$, $\Omega \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$. Let $0 < \gamma + m\beta \leq \frac{n}{(p_1)_+}$, and define $p_2(\cdot)$ by : $\frac{1}{p_1(x)} - \frac{1}{p_2(x)} = \frac{\gamma+m\beta}{n}$. Then for all $f \in L^{p_1(\cdot)}(\mathbb{R}^n)$, we have*

$$\|[b^m, T_{\Omega,\gamma}]f\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \leq C\|b\|^m_{Lip_\beta}\|f\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}.$$

**Lemma 2.5.** (1) *Suppose that $0 < \gamma < n, p_1(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$, $\Omega \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$. Let $0 < \gamma \leq \frac{n}{(p_1)_+}$ and define $p_2(\cdot)$ by $\frac{1}{p_1(x)} - \frac{1}{p_2(x)} = \frac{\gamma}{n}$, then for all $f \in L^{p_1(\cdot)}(\mathbb{R}^n)$, we have*

$$\|\mu_{\Omega,\gamma}f\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}.$$

(2) *Suppose that* $b \in Lip(\mathbb{R}^n)$, $0 < \gamma < n, p_1(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$, $\Omega \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$. *Let* $0 < \gamma + m\beta \leq \frac{n}{(p_1)_+}$ *and define* $p_2(\cdot)$ *by* $\frac{1}{p_1(x)} - \frac{1}{p_2(x)} = \frac{\gamma+m\beta}{n}$, *then for all* $f \in L^{p_1(\cdot)}(\mathbb{R}^n)$, *we have*

$$\|[b^m, \mu_{\Omega,\gamma}]f\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \leq C\|b\|^m_{Lip_\beta}\|f\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}.$$

*Proof.* (1) Applying the Minkowski inequality, we get

$$\begin{aligned}\mu_{\Omega,\gamma}(f)(x) &\leq C\int_{\mathbb{R}^n} \frac{|\Omega(x,x-y)|}{|x-y|^{n-\gamma-1}}|f(y)|\left(\int_{|x-y|}^{\infty}\frac{dt}{t^3}\right)^{\frac{1}{2}}dy\\ &\leq C\int_{\mathbb{R}^n} \frac{|\Omega(x,x-y)|}{|x-y|^{n-\gamma-1}}|f(y)|\frac{1}{|x-y|}dy\\ &\leq C\int_{\mathbb{R}^n} \frac{|\Omega(x,x-y)|}{|x-y|^{n-\gamma}}|f(y)|dy.\end{aligned}$$

We know that $T_{|\Omega|,\gamma}|f|$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ (Lemma 2.3), then we obtain

$$\begin{aligned}\|\mu_{\Omega,\gamma}(f)(x)\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} &\leq C\|T_{|\Omega|,\gamma}|f|\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\\ &\leq C\|f\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

(2) By the same way which was used in above and applying Lemma 2.4 we can easily obtain the boundedness of the commutator of fractional Marcinkiewicz integral with variable kernel on Lebesgue spaces with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$.

□

**Lemma 2.6.** [21] *If* $0 < \gamma < n$ *and* $p(\cdot), q(\cdot) \in \widetilde{B}$ *such that* $p^+ < \frac{n}{\gamma}$ *and define*

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\gamma}{n},\ x \in \mathbb{R}^n,$$

*we have*

$$\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \sim |B|^{-\frac{\gamma}{n}}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

**Lemma 2.7.** [8] *If* $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$, *then there exist constants* $\delta_1, \delta_2, C > 0$ *such that for all balls* $B$ *in* $\mathbb{R}^n$ *and all measurable subset* $S \subset B$

$$\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leq C\frac{|B|}{|S|},\ \frac{\|\chi_S\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|S|}{|B|}\right)^{\delta_1},\ \frac{\|\chi_S\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|S|}{|B|}\right)^{\delta_2}.$$

**Lemma 2.8.** [6] *If* $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$, *there exists constant* $C > 0$ *such that for any balls* $B$ *in* $\mathbb{R}^n$, *we have*

$$\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq C.$$

**Lemma 2.9.** [7] *Let $b \in Lip_\beta(\mathbb{R}^n)$, $m$ be a positive integer, and there exist constants $C > 0$, such that for any $l, j \in \mathbb{Z}$ with $l > j$*

(1) $C^{-1}\|b\|_{Lip_\beta}^m \le |B|^{-m\beta/n}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1}\|(b-b_B)^m\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|b\|_{Lip_\beta}^m$

(2) $\|(b-b_{B_j})^m\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C|B_k|^{m\beta/n}\|b\|_{Lip_\beta}^m\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

## 3. Main Theorems and their proofs

**Theorem 3.1.** *Suppose that $0 < \gamma < n, 0 < q \le \infty$ and $\Omega \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$ $(1 < r \le \infty)$. Let $p_1(\cdot), p_2(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$, $1 \le r' < (p_1)_- \le (p_1)_+ < \frac{n}{\gamma}$, $\frac{1}{p_1(x)} - \frac{1}{p_2(x)} = \frac{\gamma}{n}$ and let $\alpha(x) \in L^\infty(\mathbb{R}^n)$ be log-Hölder continuous both at the origin and at infinity, such that*

$$\lambda - n\delta_2 + \gamma < \alpha_- \le \alpha_+ < n\delta_1 - \frac{n}{r}, \tag{7}$$

*where $\delta_1, \delta_2$ are the constants in Lemma 2.7. Then $\mu_{\Omega,\gamma}$ is bounded from $M\dot{K}_{q_1,p_1(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)$ to $M\dot{K}_{q_2,p_2(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)$.*

*Proof.* If $f \in M\dot{K}_{q_1,p_1(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)$, we applying

$$\left[\sum_{j=1}^{\infty} a_j\right]^{q_1/q_2} \le \sum_{j=1}^{\infty} a_j^{q_1/q_2}, \quad a_1, a_2.... \ge 0.$$

$$\begin{aligned}\|\mu_{\Omega,\gamma}f_j\|_{M\dot{K}_{q_2,p_2(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}^{q_1} &= \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1}\left\{\sum_{k=-\infty}^{k_0}\left\|2^{k\alpha(\cdot)}\mu_{\Omega,\gamma}(f)\chi_k\right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}^{q_2}\right\}^{q_1/q_2}\\ &\le \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0}\left\|2^{k\alpha(\cdot)}\mu_{\Omega,\gamma}(f)\chi_k\right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}^{q_1}.\end{aligned}$$

If we denote

$$f(x) = \sum_{j=-\infty}^{\infty} f(x)\chi_j = \sum_{j=-\infty}^{\infty} f_j(x).$$

Then, we have

$$\begin{aligned}&\|\mu_{\Omega,\gamma}f_j\|_{M\dot{K}_{q_2,p_2(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}^{q_1}\\ &\le \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0}\left\|2^{k\alpha(\cdot)}\left(\sum_{j=-\infty}^{\infty}|\mu_{\Omega,\gamma}f_j|\right)\chi_k\right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}^{q_1}\\ &\le \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0}\left\|2^{k\alpha(\cdot)}\left(\sum_{j=-\infty}^{k-2}|\mu_{\Omega,\gamma}f_j|\right)\chi_k\right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}^{q_1}\end{aligned}$$

$$+\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \left\| 2^{k\alpha(\cdot)} \left( \sum_{j=k-1}^{k+1} |\mu_{\Omega,\gamma} f_j| \right) \chi_k \right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}^{q_1}$$

$$+\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \left\| 2^{k\alpha(\cdot)} \left( \sum_{j=k+2}^{\infty} |\mu_{\Omega,\gamma} f_j| \right) \chi_k \right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}^{q_1},$$

$$Ł_{11} + Ł_{12} + Ł_{13}.$$

Therefore, we have

$$\|\mu_{\Omega,\gamma} f_j\|_{M\dot{K}_{q_2,p_2(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}^{q_1} \le Ł_{11} + Ł_{12} + Ł_{13} \tag{8}$$

Below, we first consider $L_{12}$. By Lemma 2.1, we get

$$L_{12} = max\left\{ \sup_{k_0<0,k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \left\| 2^{k\alpha(0)} \left( \sum_{j=k-1}^{k+1} |\mu_{\Omega,\gamma} f_j| \right) \chi_k \right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}^{q_1}, \right.$$

$$\sup_{k_0\ge 0,k_0\in\mathbb{Z}} \left[ 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{-1} \left\| 2^{k\alpha(0)} \left( \sum_{j=k-1}^{k+1} |\mu_{\Omega,\gamma} f_j| \right) \chi_k \right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}^{q_1} \right.$$

$$\left.\left. +2^{-k_0\lambda q_1} \sum_{k=0}^{k_0} \left\| 2^{k\alpha_\infty} \left( \sum_{j=k-1}^{k+1} |\mu_{\Omega,\gamma} f_j| \right) \chi_k \right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}^{q_1} \right] \right\}.$$

Noting that $\mu_{\Omega,\gamma}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ (Lemma 2.5), we have

$$\begin{aligned}
\left\| 2^{k\alpha(0)} \left( \sum_{j=k-1}^{k+1} |\mu_{\Omega,\gamma} f_j| \right) \chi_k \right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}^{q_1} &\le \left\| 2^{k\alpha(0)} \left( \sum_{j=k-1}^{k+1} |\mu_{\Omega,\gamma} f_j| \right) \right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}^{q_1} \\
&\le \sum_{j=k-1}^{k+1} \|2^{k\alpha(0)} |\mu_{\Omega,\gamma} f_j|\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}^{q_1} \\
&\le \sum_{j=k-1}^{k+1} \|2^{k\alpha(0)} |f_j|\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1}.
\end{aligned}$$

Thus, we have

$$Ł_{12} \le max\left\{ C \sup_{k_0<0,k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \left\| 2^{k\alpha(0)} |f\chi_k| \right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1}, \right.$$

$$C \sup_{k_0\ge 0,k_0\in\mathbb{Z}} \left[ 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{-1} \left\| 2^{k\alpha(0)} |f\chi_k| \right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \right]$$

$$+\left[2^{-k_0\lambda q_1}\sum_{k=0}^{k_0}\left\|2^{k\alpha_\infty}|f\chi_k|\right\|^{q_1}_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right]\Bigg\}$$
$$\le C\|f\|^{q_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(\mathbb{R}^n)}. \tag{9}$$

Now we consider Ł$_{11}$, we have

$$|\mu_{\Omega,\gamma}f_j(x)|\le\left(\int_0^{|x|}\left|\int_{|x-y|\le t}\frac{\Omega(x,x-y)}{|x-y|^{n-\gamma-1}}f_j(y)dy\right|^2\frac{dt}{t^3}\right)^{\frac12}$$
$$+\left(\int_{|x|}^{\infty}\left|\int_{|x-y|\le t}\frac{\Omega(x,x-y)}{|x-y|^{n-\gamma-1}}f_j(y)dy\right|^2\frac{dt}{t^3}\right)^{\frac12}$$
$$=I_1+I_2.$$

Noting that $x\in C_k,\ j\le k-2$, then $|x-y|\sim|x|\sim|2^k|$. Hence

$$\left|\frac{1}{|x-y|^2}-\frac{1}{|x|^2}\right|\le C\frac{|y|}{|x-y|^3}. \tag{10}$$

By (10), and the Minkowski inequality, we show that

$$I_1\le C\int_{\mathbb{R}^n}\frac{|\Omega(x,x-y)|}{|x-y|^{n-\gamma-1}}|f_j(y)|\left(\int_{|x-y|}^{|x|}\frac{dt}{t^3}\right)^{\frac12}dy$$
$$\le C\int_{\mathbb{R}^n}\frac{|\Omega(x,x-y)|}{|x-y|^{n-\gamma-1}}|f_j(y)|\left|\frac{1}{|x-y|^2}-\frac{1}{|x|^2}\right|^{\frac12}dy$$
$$\le C\int_{\mathbb{R}^n}\frac{|\Omega(x,x-y)|}{|x-y|^{n-\gamma-1}}|f_j(y)|\frac{|y|^{\frac12}}{|x-y|^{\frac32}}dy$$
$$\le C2^{(j-k)\frac12}\int_{C_j}\frac{|\Omega(x,x-y)|}{|x-y|^{n-\gamma}}|f_j(y)|dy$$
$$\le C\int_{C_j}\frac{|\Omega(x,x-y)|}{|x-y|^{n-\gamma}}|f_j(y)|dy.$$

Similarly, we estimate $I_2$. By the Minkowski inequality, we get

$$I_2\le C\int_{\mathbb{R}^n}\frac{|\Omega(x,x-y)|}{|x-y|^{n-\gamma-1}}|f_j(y)|\left(\int_{|x|}^{\infty}\frac{dt}{t^3}\right)^{\frac12}dy$$

$$\leq C\int\limits_{\mathbb{R}^n}\frac{|\Omega(x,x-y)|}{|x-y|^{n-\gamma-1}}|f_j(y)|\frac{1}{|x|}dy$$
$$\leq C\int\limits_{C_j}\frac{|\Omega(x,x-y)|}{|x-y|^{n-\gamma}}|f_j(y)|dy.$$

So, we obtain that

$$|\mu_{\Omega,\gamma}f_j(x)|\leq C\int\limits_{C_j}\frac{|\Omega(x,x-y)|}{|x-y|^{n-\gamma}}|f_j(y)|dy.$$

By the generalized Hölder's inequality assures that

$$|\mu_{\Omega,\gamma}f_j(x)|\leq C2^{-k(n-\gamma)}\|\Omega(x,x-y)\|_{L^r(\mathbb{R}^n)}\|f_j(y)\|_{L^{r'}(\mathbb{R}^n)}. \tag{11}$$

Hence, for any $j\leq k-2$ and $\Omega\in L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})$, we have

$$\|\Omega(x,x-y)\|_{L^r(\mathbb{R}^n)}\leq\left[\int\limits_{2^{k-2}}^{2^k}r^{n-1}dr(\int_{s^{n-1}}|\Omega(x,y')|^r d\sigma(y')\right]^{\frac{1}{r}}$$
$$\leq C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}2^{\frac{kn}{r}}\leq C2^{\frac{kn}{r}}. \tag{12}$$

Again applying the generalized Hölder's inequality, we have

$$\|f_j(y)\|_{L^{r'}(\mathbb{R}^n)}\leq|B_j|^{-\frac{1}{r}}\|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}.$$

We can define a variable exponent $\widetilde{p_1}(\cdot)>0$ by $\frac{1}{r'}=\frac{1}{p_1(\cdot)}+\frac{1}{\widetilde{p_1}(\cdot)}$ due to $1\leq r'<p^-$, we have

$$\|f_j(y)\|_{L^{r'}(\mathbb{R}^n)}\leq C\|f_j\|_{L^{p_1(\cdot)}}\|\chi_{B_j}\|_{L^{\widetilde{p_1}(\cdot)}(\mathbb{R}^n)}. \tag{13}$$

Moreover, we can observe that $\frac{1}{\widetilde{p_1}(\cdot)}=\frac{1}{r'}-\frac{1}{p_1(\cdot)}=\frac{1}{p_1'(\cdot)}-\frac{1}{r}$, applying Lemma 2.6, we have

$$\|\chi_{B_j}\|_{L^{\widetilde{p_1}(\cdot)}(\mathbb{R}^n)}\sim|B_j|^{\frac{-1}{r}}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}. \tag{14}$$

By (11)-(14), it follows that

$$|\mu_{\Omega,\gamma}f_j(x)|\leq C2^{-k(n-\gamma)}2^{(k-j)\frac{n}{r}}\|f_j\|_{L^{p_1(\cdot)}}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}.$$

Thus, we obtain

$$\left\|2^{k\alpha(\cdot)}\sum_{j=-\infty}^{k-2}|\mu_{\Omega,\gamma}f_j|\chi_k\right\|_{L^{p_2(\cdot)}}$$
$$\leq C\sum_{j=-\infty}^{k-2}2^{-k(n-\gamma)}2^{(k-j)\frac{n}{r}}2^{(k-j)\alpha_+}\|2^{j\alpha(\cdot)}f_j\|_{L^{p_1(\cdot)}}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}\|\chi_{B_k}\|_{L^{p_2(\cdot)}}.$$

Noting that, if $\frac{1}{p_1(\cdot)}-\frac{1}{p_2(\cdot)}=\frac{\gamma}{n}$, then

$$C_1|B|^{\frac{\gamma}{n}}\|\chi_B\|_{L^{p_2(\cdot)}}\leq\|\chi_B\|_{L^{p_1(\cdot)}}\leq C_2|B|^{\frac{\gamma}{n}}\|\chi_B\|_{L^{p_2(\cdot)}}$$

(see[22], p.370).
From this, and using Lemmas 2.7-2.8, we have

$$\leq \sum_{j=-\infty}^{k-2} 2^{-k(n-\gamma)}2^{(k-j)\frac{n}{r}}2^{(k-j)\alpha_+}\|2^{j\alpha(\cdot)}f_j\|_{L^{p_1(\cdot)}}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}2^{-k\gamma}\|\chi_{B_k}\|_{L^{p_1(\cdot)}}$$
$$\leq \sum_{j=-\infty}^{k-2} 2^{-k(n-\gamma)}2^{(k-j)\frac{n}{r}}2^{(k-j)\alpha_+}\|2^{j\alpha(\cdot)}f_j\|_{L^{p_1(\cdot)}}2^{k(n-\gamma)}\frac{\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}}{\|\chi_{B_k}\|_{L^{p_1'(\cdot)}}}$$
$$\leq C\sum_{j=-\infty}^{k-2} 2^{(j-k)(n\delta_1-\frac{n}{r}-\alpha_+)}\|2^{j\alpha(\cdot)}f_j\|_{L^{p_1(\cdot)}}$$
$$\leq C\sum_{j=-\infty}^{k-2} 2^{(j-k)\xi}\|2^{j\alpha(\cdot)}f_j\|_{L^{p_1(\cdot)}}. \tag{15}$$

Where $\xi=(n\delta_1-\frac{n}{r}-\alpha_+)$, from condition (7) noting that $\xi>0$. Now we consider two cases $1<q_1<\infty$ and $0<q_1\leq 1$.
If $0<q_1\leq 1$, we have

$$\begin{aligned} L_{11} &\leq C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0}\sum_{j=-\infty}^{k-2} 2^{(j-k)\xi q_1}\|2^{j\alpha(\cdot)}f_j\|^{q_1}_{L^{p_1(\cdot)}} \\ &\leq C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1}\sum_{j=-\infty}^{k_0-2}\|2^{j\alpha(\cdot)}f_j\|^{q_1}_{L^{p_1(\cdot)}}\sum_{k=j+2}^{k_0}2^{(j-k)\xi q_1} \\ &\leq C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1}\sum_{j=-\infty}^{k_0-2}\|2^{j\alpha(\cdot)}f_j\|^{q_1}_{L^{p_1(\cdot)}} \\ &\leq C\|f\|^{q_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{16}$$

Case 2: If $1<q_1<\infty$, applying Hölder's inequality, we obtain that

$$\begin{aligned} L_{11} &\leq C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0}\left(\sum_{j=-\infty}^{k-2} 2^{(j-k)\xi q_1/2}\|2^{j\alpha(\cdot)}f_j\|^{q_1}_{L^{p_1(\cdot)}}\right) \\ &\quad\cdot\left(\sum_{j=-\infty}^{k-2} 2^{(j-k)\xi q_1'/2}\right)^{q_1/q_1'} \\ &\leq C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0}\left(\sum_{j=-\infty}^{k-2} 2^{(j-k)\xi q_1/2}\|2^{j\alpha(\cdot)}f_j\|^{q_1}_{L^{p_1(\cdot)}}\right) \\ &\leq C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1}\sum_{j=-\infty}^{k_0-2}\|2^{j\alpha(\cdot)}f_j\|^{q_1}_{L^{p_1(\cdot)}}\sum_{k=j+2}^{k_0}2^{(j-k)\xi q_1/2} \end{aligned}$$

$$\le \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1}\sum_{j=-\infty}^{k_0-2}\|2^{j\alpha(\cdot)}f_j\|_{L^{p_1(\cdot)}}^{q_1}$$
$$\le C\|f\|^{q_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(\mathbb{R}^n)}. \tag{17}$$

Finally, we consider $L_3$. Again by Lemma 2.1 we have

$$L_{13}\approx max\left\{\sup_{k_0<0,k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0}\left\|2^{k\alpha(0)}\left(\sum_{j=k+2}^{\infty}|\mu_{\Omega,\gamma}f_j|\right)\chi_k\right\|^{q_1}_{L^{p_2(\cdot)}(\mathbb{R}^n)},\right.$$
$$\sup_{k_0\ge0,k_0\in\mathbb{Z}}\left[2^{-k_0\lambda q_1}\sum_{k=-\infty}^{-1}\left\|2^{k\alpha(0)}\left(\sum_{j=k+2}^{\infty}|\mu_{\Omega,\gamma}f_j|\right)\chi_k\right\|^{q_1}_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right.$$
$$\left.\left.+2^{-k_0\lambda q_1}\sum_{k=0}^{k_0}\left\|2^{k\alpha_\infty}\left(\sum_{j=k+2}^{\infty}|\mu_{\Omega,\gamma}f_j|\right)\chi_k\right\|^{q_1}_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right]\right\}$$
$$= max\{F,G\}. \tag{18}$$

We consider F. Let $x\in C_k, j\ge k+2$, then $|x-y|\sim|y|$, we have

$$|\mu_{\Omega,\gamma}f_j(x)|\le\left(\int_0^{|y|}\left|\int_{|x-y|\le t}\frac{\Omega(x,x-y)}{|x-y|^{n-\gamma-1}}f_j(y)dy\right|^2\frac{dt}{t^3}\right)^{\frac12}$$
$$+\left(\int_{|y|}^{\infty}\left|\int_{|x-y|\le t}\frac{\Omega(x,x-y)}{|x-y|^{n-\gamma-1}}f_j(y)dy\right|^2\frac{dt}{t^3}\right)^{\frac12}$$
$$= I_3+I_4.$$

By (10), and the Minkowski inequality, we show that

$$I_3\le C\int_{\mathbb{R}^n}\frac{|\Omega(x,x-y)|}{|x-y|^{n-\gamma-1}}|f_j(y)|\left(\int_{|x-y|}^{|y|}\frac{dt}{t^3}\right)^{\frac12}dy$$
$$\le C\int_{\mathbb{R}^n}\frac{|\Omega(x,x-y)|}{|x-y|^{n-\gamma-1}}|f_j(y)|\left|\frac{1}{|x-y|^2}-\frac{1}{|y|^2}\right|^{\frac12}dy$$
$$\le C\int_{\mathbb{R}^n}\frac{|\Omega(x,x-y)|}{|x-y|^{n-\gamma-1}}|f_j(y)|\frac{|x|^{\frac12}}{|x-y|^{\frac32}}dy$$
$$\le C2^{(k-j)\frac12}\int_{C_j}\frac{|\Omega(x,x-y)|}{|x-y|^{n-\gamma}}|f_j(y)|dy$$

$$\leq C\int\limits_{C_j}\frac{|\Omega(x,x-y)|}{|x-y|^{n-\gamma}}|f_j(y)|dy.$$

Similarly, we estimate $I_4$. By the Minkowski inequality and, we get

$$\begin{aligned}
I_4 &\leq C\int\limits_{\mathbb{R}^n}\frac{|\Omega(x,x-y)|}{|x-y|^{n-\gamma-1}}|f_j(y)|\left(\int\limits_{|y|}^{\infty}\frac{dt}{t^3}\right)^{\frac{1}{2}}dy\\
&\leq C\int\limits_{\mathbb{R}^n}\frac{|\Omega(x,x-y)|}{|x-y|^{n-\gamma-1}}|f_j(y)|\frac{1}{|y|}dy\\
&\leq C\int\limits_{C_j}\frac{|\Omega(x,x-y)|}{|x-y|^{n-\gamma}}|f_j(y)|dy.
\end{aligned}$$

So, we have

$$|\mu_{\Omega,\gamma}f_j(x)|\leq C\int\limits_{C_j}\frac{|\Omega(x,x-y)|}{|x-y|^{n-\gamma}}|f_j(y)|dy.$$

Applying the generalized Hölder's inequality assures that

$$|\mu_{\Omega,\gamma}f_j(x)|\leq C2^{-j(n-\gamma)}\|\Omega(x,x-y)\|_{L^r(\mathbb{R}^n)}\|f_j(y)\|_{L^{r'}(\mathbb{R}^n)}. \tag{19}$$

Hence, for any $j\geq k+2$ and $\Omega\in L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})$, we can easy get

$$\|\Omega(x,x-y)\|_{L^r(\mathbb{R}^n)}\leq C2^{\frac{jn}{r}}\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}\leq C2^{\frac{jn}{r}}. \tag{20}$$

The same way which was used in (13) and (14), we have

$$\|f_j(y)\|_{L^{r'}(\mathbb{R}^n)}\leq C|B_j|^{-\frac{1}{r}}2^{-j(n-\gamma)}\|f_j\|_{L^{p_1(\cdot)}}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}. \tag{21}$$

Combining (20) with (21), we obtain that

$$|\mu_{\Omega,\gamma}f_j(x)|\leq C2^{-j(n-\gamma)}\|f_j\|_{L^{p_1(\cdot)}}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}. \tag{22}$$

By (22) and again applying Lemmas 2.7-2.8, we show that

$$\begin{aligned}
\left\|\sum_{j=k+2}^{\infty}|\mu_{\Omega,\gamma}f_j|\chi_k\right\|_{L^{p_2(\cdot)}} &\leq C\sum_{j=k+2}^{\infty}2^{-j(n-\gamma)}\|f_j\|_{L^{p_1(\cdot)}}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}\|\chi_{B_k}\|_{L^{p_2(\cdot)}}\\
&\leq C\sum_{j=k+2}^{\infty}2^{-j(n-\gamma)}\|f_j\|_{L^{p_1(\cdot)}}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}2^{-k\gamma}\|\chi_{B_k}\|_{L^{p_1(\cdot)}}\\
&\leq C\sum_{j=k+2}^{\infty}2^{(j-k)\gamma}\|f_j\|_{L^{p_1(\cdot)}}\frac{\|\chi_{B_k}\|_{L^{p_1(\cdot)}}}{\|\chi_{B_j}\|_{L^{p_1(\cdot)}}}\\
&\leq C\sum_{j=k+2}^{\infty}2^{(j-k)\gamma}\|f_j\|_{L^{p_1(\cdot)}}2^{n\delta_2(k-j)}
\end{aligned}$$

$$
\begin{aligned}
&\leq C \sum_{j=k+2}^{\infty} 2^{(k-j)(n\delta_2-\gamma)} \|f_j\|_{L^{p_1(\cdot)}} \\
&\leq C \sum_{j=k+2}^{\infty} 2^{(k-j)\sigma} \|f_j\|_{L^{p_1(\cdot)}}. \qquad (23)
\end{aligned}
$$

Where $\sigma = (n\delta_2 - \gamma)$. Now also we have two cases $1 < q_1 < \infty$ and $0 < q_1 \leq 1$. If $0 < q_1 \leq 1$, by (23) and $\sigma + \alpha(0) > \sigma + \alpha_-$, we obtain that

$$
\begin{aligned}
F &\leq C \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q_1} \sum_{j=k+2}^{\infty} 2^{(k-j)\sigma q_1} \|f_j\|_{L^{p_1(\cdot)}}^{q_1} \\
&\leq C \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q_1} \sum_{j=k+2}^{k_0-1} 2^{(k-j)\sigma q_1} \|f_j\|_{L^{p_1(\cdot)}}^{q_1} \\
&\quad + \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q_1} \sum_{j=k_0}^{\infty} 2^{(k-j)\sigma q_1} \|f_j\|_{L^{p_1(\cdot)}}^{q_1} \\
&= F_1 + F_2.
\end{aligned}
$$

For $F_1$, we have

$$
\begin{aligned}
F_1 &\leq C \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{j=-\infty}^{k_0-1} 2^{j\alpha(0)q_1} \|f_j\|_{L^{p_1(\cdot)}}^{q_1} \sum_{k=-\infty}^{j-2} 2^{(k-j)(\sigma+\alpha(0))q_1} \\
&\leq C \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{j=-\infty}^{k_0-1} 2^{j\alpha(0)q_1} \|f_j\|_{L^{p_1(\cdot)}}^{q_1} \\
&\leq C \|f\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}^{q_1}. \qquad (24)
\end{aligned}
$$

For $F_2$, since $\lambda - \sigma - \alpha(0) < \lambda - \sigma - \alpha_-$, we obtain that

$$
\begin{aligned}
F_2 &\leq C \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q_1} \sum_{j=k_0}^{\infty} 2^{(k-j)\sigma q_1} 2^{-j\alpha(0)q_1} 2^{j\lambda q_1} \\
&\qquad \times 2^{-j\lambda q_1} \sum_{m=-\infty}^{j} 2^{m\alpha(0)q_1} \|f_m\|_{L^{p_1(\cdot)}}^{q_1} \\
&\leq C \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \left( \sum_{k=-\infty}^{k_0} 2^{k(\sigma+\alpha(0))q_1} \right) \\
&\quad \times \left( \sum_{j=k_0}^{\infty} 2^{j(\lambda-\sigma-\alpha(0))q_1} \right) \|f\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}^{q_1} \\
&\leq C \|f\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}^{q_1}. \qquad (25)
\end{aligned}
$$

Case 2: If $1 < q_1 < \infty$, we get

$$
\begin{aligned}
F &\le C \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q_1} \left( \sum_{j=k+2}^{k_0} 2^{(k-j)\sigma\|f_j\|_{L^{p_1(\cdot)}}} \right)^{q_1} \\
&+ \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q_1} \left( \sum_{j=k_0+1}^{\infty} 2^{(k-j)\sigma\|f_j\|_{L^{p_1(\cdot)}}} \right)^{q_1} \\
&= F_3 + F_4.
\end{aligned}
$$

For $F_3$, using Hölder's inequality, we have

$$
\begin{aligned}
F_3 &\le C \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{j=k+2}^{k_0} 2^{j\alpha(0)q_1} \|f_j\|_{L^{p_1(\cdot)}} 2^{(k-j)(\sigma+\alpha(0))q_1/2} \\
&\times \left( \sum_{j=k+2}^{k_0} 2^{(k-j)(\sigma+\alpha(0))q_1'/2} \right)^{q_1/q_1'} \\
&\le C \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{j=k+2}^{k_0} 2^{j\alpha(0)q_1} \|f_j\|_{L^{p_1(\cdot)}} 2^{(k-j)(\sigma+\alpha(0))q_1/2} \\
&\le C \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{j=-\infty}^{k_0} 2^{j\alpha(0)q_1} \|f_j\|_{L^{p_1(\cdot)}} \sum_{k=-\infty}^{j-2} 2^{(k-j)(\sigma+\alpha(0))q_1/2} \\
&\le C\|f\|^{q_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(\mathbb{R}^n)}. \qquad (26)
\end{aligned}
$$

For $F_4$, we have

$$
\begin{aligned}
F_4 &\le C \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \\
&\times \left( \sum_{j=k_0+1}^{\infty} 2^{j\alpha(0)} \|f_j\|_{L^{p_1(\cdot)}} 2^{(k-j)(\sigma+\alpha(0)+\lambda)/2} 2^{(k-j)(\sigma+\alpha(0)-\lambda)/2} \right)^{q_1} \\
&\le C \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{j=k_0+1}^{\infty} 2^{j\alpha(0)} \|f_j\|_{L^{p_1(\cdot)}} 2^{(k-j)(\sigma+\alpha(0)+\lambda)q_1/2} \\
&\times \left( \sum_{j=k_0+1}^{\infty} 2^{(k-j)(\sigma+\alpha(0)-\lambda)q_1'/2} \right)^{q_1/q_1'} \\
&\le C \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{j=k_0+1}^{\infty} 2^{j\alpha(0)} \|f_j\|_{L^{p_1(\cdot)}} 2^{(k-j)(\sigma+\alpha(0)+\lambda)q_1/2}
\end{aligned}
$$

$$\leq C \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{j=k_0+1}^{\infty} 2^{(k-j)(\sigma+\alpha(0)+\lambda)q_1/2} 2^{j\lambda q_1} \times 2^{-j\lambda q_1}$$

$$\times \sum_{m=-\infty}^{j} 2^{m\alpha(0)q_1} \|f_m\|_{L^{p_1(\cdot)}}^{q_1}$$

$$\leq C \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} 2^{k\lambda q_1} \sum_{j=k_0+1}^{\infty} 2^{(k-j)(\sigma+\alpha(0)-\lambda)q_1/2} \|f\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}^{q_1}$$

$$\leq C\|f\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}^{q_1}. \tag{27}$$

We can omit the estimate of $G$ since it is essentially similar to that of $F$. Hence the proof of Theorem 3.1 is completed. □

**Theorem 3.2.** *Suppose that $b \in Lip_\beta(\mathbb{R}^n)$, $m \in \mathbb{N}$, $0 < \gamma < n, 0 < q \leq \infty$, and $\Omega \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$ $(1 < r \leq \infty)$. Let $p_1(\cdot), p_2(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$, $1 \leq r' < (p_1)_- \leq (p_1)_+ < \frac{n}{\gamma+m\beta}$, $\frac{1}{p_1(x)} - \frac{1}{p_2(x)} = \frac{\gamma+m\beta}{n}$ and let $\alpha(x) \in L^\infty(\mathbb{R}^n)$ be log-Hölder continuous both at the origin and at infinity and satisfying (1.7). Then $[b^m, \mu_{\Omega,\gamma}]$ is bounded from $M\dot{K}_{q_1,p_1(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)$ to $M\dot{K}_{q_2,p_2(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)$.*

*Proof.* If $f \in M\dot{K}_{q_1,p_1(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)$, we applying $\left[\sum_{j=1}^{\infty} a_j\right]^{q_1/q_2} \leq \sum_{j=1}^{\infty} a_j^{q_1/q_2}$, $a_1, a_2 .... \geq 0$.

$$\|[b^m, \mu_{\Omega,\gamma}]f_j\|_{M\dot{K}_{q_2,p_2(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}^{q_1}$$

$$= \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \left\{ \sum_{k=-\infty}^{k_0} \left\| 2^{k\alpha(\cdot)} [b^m, \mu_{\Omega,\gamma}](f)\chi_k \right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}^{q_2} \right\}^{q_1/q_2}$$

$$\leq \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \left\| 2^{k\alpha(\cdot)} [b^m, \mu_{\Omega,\gamma}](f)\chi_k \right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}^{q_1}.$$

If we denote

$$f(x) = \sum_{j=-\infty}^{\infty} f(x)\chi_j = \sum_{j=-\infty}^{\infty} f_j(x).$$

Then we have

$$\|[b^m, \mu_{\Omega,\gamma}]f_j\|_{M\dot{K}_{q_2,p_2(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}^{q_1}$$

$$\leq \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \left\| 2^{k\alpha(\cdot)} \left( \sum_{j=-\infty}^{\infty} |[b^m, \mu_{\Omega,\gamma}]f_j| \right) \chi_k \right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}^{q_1}$$

$$
\begin{aligned}
&\leq \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \left\| 2^{k\alpha(\cdot)} \left( \sum_{j=-\infty}^{k-2} |[b^m,\mu_{\Omega,\gamma}]f_j| \right) \chi_k \right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}^{q_1} \\
&\quad + \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \left\| 2^{k\alpha(\cdot)} \left( \sum_{j=k-1}^{k+1} |[b^m,\mu_{\Omega,\gamma}]f_j| \right) \chi_k \right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}^{q_1} \\
&\quad + \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \left\| 2^{k\alpha(\cdot)} \left( \sum_{j=k+2}^{\infty} |[b^m,\mu_{\Omega,\gamma}]f_j| \right) \chi_k \right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}^{q_1} \\
&= L_{21} + L_{22} + L_{23}.
\end{aligned}
$$

First, we consider $L_{22}$. Noting that $[b^m,\mu_{\Omega,\gamma}]$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ (Lemmas 2.5), as argued about Ł$_{12}$ in the proof of Theorem 3.1, immediately get

$$
Ł_{22} \leq C\|b\|_{Lip_\beta(\mathbb{R}^n)}^{mq_1} \|f\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}^{q_1}. \tag{28}
$$

Now we consider $L_{21}$, we have

$$
\begin{aligned}
|[b^m,\mu_{\Omega,\gamma} f(x)]| &\leq \left( \int_0^{|x|} \left| \int_{|x-y|\leq t} \frac{\Omega(x,x-y)}{|x-y|^{n-\gamma-1}} [b(x)-b(y)] f_j(y)dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}} \\
&\quad + \left( \int_{|x|}^{\infty} \left| \int_{|x-y|\leq t} \frac{\Omega(x,x-y)}{|x-y|^{n-\gamma-1}} [b(x)-b(y)] f_j(y)dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}} \\
&= J_1 + J_2.
\end{aligned}
$$

Noting that $x\in C_k$, $j\leq k-2$, then $|x-y|\sim|x|\sim|2^k|$. By (10) and Minkowski inequality, we show that

$$
\begin{aligned}
J_1 &\leq C\int_{\mathbb{R}^n} \frac{|\Omega(x,x-y)|}{|x-y|^{n-\gamma-1}} |b(x)-b(y)|^m |f_j(y)| \left( \int_{|x-y|}^{|x|} \frac{dt}{t^3} \right)^{\frac{1}{2}} dy \\
&\leq C\int_{\mathbb{R}^n} \frac{|\Omega(x,x-y)|}{|x-y|^{n-\gamma-1}} |b(x)-b(y)|^m |f_j(y)| \left| \frac{1}{|x-y|^2} - \frac{1}{|x|^2} \right|^{\frac{1}{2}} dy \\
&\leq C\int_{\mathbb{R}^n} \frac{|\Omega(x,x-y)|}{|x-y|^{n-\gamma-1}} |b(x)-b(y)|^m |f_j(y)| \frac{|y|^{\frac{1}{2}}}{|x-y|^{\frac{3}{2}}} dy \\
&\leq C2^{(j-k)\frac{1}{2}} \int_{C_j} \frac{|\Omega(x,x-y)|}{|x-y|^{n-\gamma}} |b(x)-b(y)|^m |f_j(y)| dy
\end{aligned}
$$

$$\leq C \int\limits_{C_j} \frac{|\Omega(x, x-y)|}{|x-y|^{n-\gamma}} |b(x) - b(y)|^m |f_j(y)| dy.$$

Similarly, we estimate $J_1$. By the Minkowski inequality, we have

$$J_2 \leq C \int_{\mathbb{R}^n} \frac{|\Omega(x, x-y)|}{|x-y|^{n-\gamma-1}} |b(x) - b(y)|^m |f_j(y)| \left( \int\limits_{|x|}^{\infty} \frac{dt}{t^3} \right)^{\frac{1}{2}} dy$$
$$\leq C \int\limits_{C_j} \frac{|\Omega(x, x-y)|}{|x-y|^{n-\gamma-1}} |b(x) - b(y)|^m |f_j(y)| \frac{1}{|x|} dy$$
$$\leq C \int\limits_{C_j} \frac{|\Omega(x, x-y)|}{|x-y|^{n-\gamma}} |b(x) - b(y)|^m |f_j(y)| dy.$$

So, we obtain that

$$|[b^m, \mu_{\Omega,\gamma} f(x)]| \leq C \int\limits_{C_j} \frac{|\Omega(x, x-y)|}{|x-y|^{n-\gamma}} |b(x) - b(y)|^m |f_j(y)| dy$$
$$\leq C 2^{-k(n-\gamma)} \int\limits_{C_j} |\Omega(x, x-y)| |b(x) - b(y)|^m |f_j(y)| dy$$
$$\leq C 2^{-k(n-\gamma)} |b(x) - b_j|^m \int\limits_{C_j} |\Omega(x, x-y)| |f_j(y)| dy$$
$$+ 2^{-k(n-\gamma)} \int\limits_{C_j} |\Omega(x, x-y)| |b(y) - b_j|^m |f_j(y)| dy$$
$$= A_1 + A_2$$

For $A_1$, it is easy to check that

$$A_1 \leq C 2^{-k(n-\gamma)} 2^{(k-j)\frac{n}{r}} |b(x) - b_j|^m \|f_j(y)\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}.$$

Now we consider $A_2$. By the generalized Hölder's inequality, we have

$$A_2 \leq C 2^{-k(n-\gamma)} \|\Omega(x, x-y)\|_{L^r(\mathbb{R}^n)} \|f_j(y)(b(y) - b_j)^m\|_{L^{r'}(\mathbb{R}^n)}$$

Similar to (13), (14) and applying Lemma 2.9, we have

$$\|f_j(y)(b(y) - b_j)^m\|_{L^{r'}(\mathbb{R}^n)}$$
$$\leq \|f_j\|_{L^{p_1(\cdot)}} \|(b(y) - b_j)^m \chi_{B_j}\|_{L^{\widetilde{p_1}(\cdot)}}$$
$$\leq C \|b\|_{Lip_\beta}^m \|f_j\|_{L^{p_1(\cdot)}} |B_j|^{\frac{m\beta}{n}} \|\chi_{B_j}\|_{L^{\widetilde{p_1}(\cdot)}}$$
$$\leq C \|b\|_{Lip_\beta}^m |B_j|^{-\frac{1}{r}} \|f_j\|_{L^{p_1(\cdot)}} |B_j|^{\frac{m\beta}{n}} \|\chi_{B_j}\|_{L^{p_1'(\cdot)}}.$$

Thus, we obtain

$$A_2 \le C\|b\|_{Lip_\beta}^m 2^{-k(n-\gamma)} 2^{(k-j)\frac{n}{r}} \|f_j\|_{L^{p_1(\cdot)}} |B_j|^{\frac{m\beta}{n}} \|\chi_{B_j}\|_{L^{p_1'(\cdot)}}$$

From $A_1, A_2$ and again applying Lemma 2.9, we have

$$\begin{aligned}
&\left\| 2^{k\alpha(\cdot)} \sum_{j=-\infty}^{k-2} [b^m, \mu_{\Omega,\gamma}] f_j, \chi_k \right\|_{L^{p_2(\cdot)}} \\
&\le C \sum_{j=-\infty}^{k-2} 2^{-k(n-\gamma)} 2^{(k-j)\frac{n}{r}} 2^{(k-j)\alpha_+} \|2^{j\alpha(\cdot)} f_j\|_{L^{p_1(\cdot)}} \\
&\times \left[ \|(b(x)-b_j)^m \chi_k\|_{L^{p_2(\cdot)}} \|\chi_{B_j}\|_{L^{p_1'(\cdot)}} + \|b\|_{Lip_\beta}^m |B_j|^{\frac{m\beta}{n}} \|\chi_j\|_{L^{p_1'(\cdot)}} \|\chi_{B_k}\|_{L^{p_2(\cdot)}} \right] \\
&\le C\|b\|_{Lip_\beta}^m \sum_{j=-\infty}^{k-2} 2^{-k(n-\gamma)} 2^{(k-j)\frac{n}{r}} 2^{(k-j)\alpha_+} \|2^{j\alpha(\cdot)} f_j\|_{L^{p_1(\cdot)}} \\
&\times \left[ |B_k|^{\frac{m\beta}{n}} \|\chi_{B_j}\|_{L^{p_1'(\cdot)}} \|\chi_k\|_{L^{p_2(\cdot)}} + |B_j|^{\frac{m\beta}{n}} \|\chi_j\|_{L^{p_1'(\cdot)}} \|\chi_{B_k}\|_{L^{p_2(\cdot)}} \right].
\end{aligned}$$

Noting that, if $\frac{1}{p_1(\cdot)} - \frac{1}{p_2(\cdot)} = \frac{\gamma+m\beta}{n}$, then $C_1|B|^{\frac{\gamma+m\beta}{n}} \|\chi_B\|_{L^{p_2(\cdot)}} \le \|\chi_B\|_{L^{p_1(\cdot)}} \le C_2|B|^{\frac{\gamma+m\beta}{n}} \|\chi_B\|_{L^{p_2(\cdot)}}$ (see[22],p.370).
From this, and using Lemmas 2.7-2.8, we have

$$\begin{aligned}
&\left\| 2^{k\alpha(\cdot)} \sum_{j=-\infty}^{k-2} [b^m, \mu_{\Omega,\gamma}] f_j, \chi_k \right\|_{L^{p_2(\cdot)}} \\
&\le C\|b\|_{Lip_\beta}^m \sum_{j=-\infty}^{k-2} 2^{-kn} 2^{(k-j)\frac{n}{r}} 2^{(k-j)\alpha_+} \|2^{j\alpha(\cdot)} f_j\|_{L^{p_1(\cdot)}} \\
&\quad \times \left[ \|\chi_{B_j}\|_{L^{p_1'(\cdot)}} \|\chi_{B_k}\|_{L^{p_1(\cdot)}} + \frac{|B_j|^{\frac{m\beta}{n}}}{|B_k|^{\frac{m\beta}{n}}} \|\chi_j\|_{L^{p_1'(\cdot)}} \|\chi_{B_k}\|_{L^{p_1(\cdot)}} \right] \\
&\le C\|b\|_{Lip_\beta}^m \sum_{j=-\infty}^{k-2} 2^{-kn} 2^{(k-j)\frac{n}{r}} 2^{(k-j)\alpha_+} \|2^{j\alpha(\cdot)} f_j\|_{L^{p_1(\cdot)}} \|\chi_{B_j}\|_{L^{p_1'(\cdot)}} \|\chi_{B_k}\|_{L^{p_1(\cdot)}} \\
&\le C\|b\|_{Lip_\beta}^m \sum_{j=-\infty}^{k-2} 2^{-kn} 2^{(k-j)\frac{n}{r}} 2^{(k-j)\alpha_+} \|2^{j\alpha(\cdot)} f_j\|_{L^{p_1(\cdot)}} |B_k| \frac{\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}}{\|\chi_{B_k}\|_{L^{p_1'(\cdot)}}} \\
&\le C\|b\|_{Lip_\beta}^m \sum_{j=-\infty}^{k-2} 2^{(j-k)(n\delta_1 - \frac{n}{r} - \alpha_+)} \|2^{j\alpha(\cdot)} f_j\|_{L^{p_1(\cdot)}}. \qquad (29)
\end{aligned}$$

Furthermore, when $\alpha_+ < n\delta_1 - \frac{n}{r}$, then the same arguments as $L_{11}$ before, we can conclude that for all $0 < q < \infty$,

$$Ł_{21} \le C\|b\|_{Lip_\beta}^{mq_1} \|f\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}^{q_1}. \qquad (30)$$

Finally, we consider $L_{23}$. Again by Lemma 2.1, we have

$$\text{Ł}_{23} \approx max\left\{ \sup_{k_0<0,k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \left\| 2^{k\alpha(0)} \left( \sum_{j=k+2}^{\infty} |[b^m, \mu_{\Omega,\gamma}]f_j| \right) \chi_k \right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}^{q_1}, \right.$$

$$\sup_{k_0\geq 0,k_0\in\mathbb{Z}} \left[ 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{-1} \left\| 2^{k\alpha(0)} \left( \sum_{j=k+2}^{\infty} |[b^m, \mu_{\Omega,\gamma}]f_j| \right) \chi_k \right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}^{q_1} \right.$$

$$\left.\left. +2^{-k_0\lambda q_1} \sum_{k=0}^{k_0} \left\| 2^{k\alpha_\infty} \left( \sum_{j=k+2}^{\infty} |[b^m, \mu_{\Omega,\gamma}]f_j| \right) \chi_k \right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}^{q_1} \right] \right\}$$

$$= max\{L, M\}.$$

New we can choose $L$, then we have

$$|[b^m, \mu_{\Omega,\gamma}f(x)]| \leq \left( \int_0^{|y|} \left| \int_{|x-y|\leq t} \frac{\Omega(x, x-y)}{|x-y|^{n-\gamma-1}} [b(x) - b(y)]f_j(y)dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}}$$

$$+ \left( \int_{|y|}^{\infty} \left| \int_{|x-y|\leq t} \frac{\Omega(x, x-y)}{|x-y|^{n-\gamma-1}} [b(x) - b(y)]f_j(y)dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}}$$

$$= J_3 + J_4.$$

Similar to the estimates of $J_1, J_2$, such that $x \in C_k, j \geq k+2$, we have

$$\begin{aligned} |[b^m, \mu_{\Omega,\gamma}f(x)]| &\leq C \int_{C_j} \frac{|\Omega(x, x-y)|}{|x-y|^{n-\gamma}} |b(x) - b(y)|^m |f_j(y)|dy \\ &\leq C2^{-j(n-\gamma)} \int_{C_j} |\Omega(x, x-y)||b(x) - b(y)|^m |f_j(y)|dy \\ &\leq C2^{-j(n-\gamma)} |b(x) - b_j|^m \int_{C_j} |\Omega(x, x-y)||f_j(y)|dy \\ &\quad + 2^{-j(n-\gamma)} \int_{C_j} |\Omega(x, x-y)||b(y) - b_j|^m |f_j(y)|dy \\ &= A_3 + A_4. \end{aligned}$$

For $A_3$, it is easy to check that

$$A_3 \leq C2^{-j(n-\gamma)} |b(x) - b_j|^m \|f_j(y)\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}.$$

By the same way which was used in $A_2$, we obtain that

$$A_4 \le C\|b\|_{Lip_\beta}^m 2^{-j(n-\gamma)}\|f_j\|_{L^{p_1(\cdot)}}|B_j|^{\frac{m\beta}{n}}\|\chi_j\|_{L^{p_1'(\cdot)}}.$$

From $A_3$ and $A_4$, and also applying Lemmas 2.7-2.8, we show that

$$\begin{aligned}
&\left\|\sum_{j=k+2}^{\infty}[b^m,\mu_{\Omega,\gamma}]f_j,\chi_k\right\|_{L^{p_2(\cdot)}} \le C\sum_{j=k+2}^{\infty}2^{-j(n-\gamma)}\|f_j\|_{L^{p_1(\cdot)}}\\
&\quad\times\left[\|(b(x)-b_j)^m\chi_k\|_{L^{p_2(\cdot)}}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}+\|b\|_{Lip_\beta}^m|B_j|^{\frac{m\beta}{n}}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}\|\chi_{B_k}\|_{L^{p_2(\cdot)}}\right]\\
&\le C\|b\|_{Lip_\beta}^m\sum_{j=k+2}^{\infty}2^{-j(n-\gamma)}2^{-k\gamma}\|f_j\|_{L^{p_1(\cdot)}}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}\|\chi_{B_k}\|_{L^{p_1(\cdot)}}\\
&\le C\|b\|_{Lip_\beta}^m\sum_{j=k+2}^{\infty}2^{-j(n-\gamma)}2^{-k\gamma}\|f_j\|_{L^{p_1(\cdot)}}|B_j|\frac{\|\chi_{B_k}\|_{L^{p_1(\cdot)}}}{\|\chi_{B_j}\|_{L^{p_1(\cdot)}}}\\
&\le C\|b\|_{Lip_\beta}^m\sum_{j=k+2}^{\infty}2^{(k-j)(n\delta_2-\gamma)}\|f_j\|_{L^{p_1(\cdot)}}. \qquad (31)
\end{aligned}$$

Furthermore, when $\alpha_+ \le \lambda - n\delta_2 + \gamma$, the similar way to the estimate of Ł$_{13}$ before, we can easily get for all $0 < q < \infty$,

$$L_{23} \le C\|b\|_{Lip_\beta}^{mq_1}\|f\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}^{q_1}. \qquad (32)$$

We can omit the estimate of $M$ since it is essentially similar to that of $L$. Hence the proof of Theorem 3.2 is completed. □

**Remark 3.1.** If the variable exponent $\alpha(\cdot)$ is constant, our results of this paper all hold.

## Competing Interests

The authors declare that they have no competing interests.

## References

1. Marcinkiewicz, J. (1938). Sur quelques intégrales du type de Dini. *Ann. Soc. Polonaise. Math, 17*, 42-50.
2. Stein, E. M. (1958). On the functions of Littlewood-Paley, Lusin, and Marcinkiewicz. *Transactions of the American Mathematical Society, 88*(2), 430-466. https://doi.org/10.1090/S0002-9947-1958-0112932-2
3. Torchinsky, A., & Wang, S. (1990). A note on the Marcinkiewicz integral. *In Colloquium Mathematicae* (Vol. 1, No. 60-61, pp. 235-243).
4. Shao, X., Tao, S., & Wang, S. (2013). Parametric Marcinkiewicz Integrals with Variable Kernel on Weak Hardy Spaces [J]. *Mathematica Applicata, 1*, 026.
5. Kovcik, O., & Rkosnk, J. (1991). On spaces $L^{p(x)}$ and $W^{k,p(x)}$. *Czechoslovak Mathematical Journal, 41*(4), 592-618.
6. Izuki, M. (2010). Boundedness of commutators on Herz spaces with variable exponent. *Rendiconti del Circolo Matematico di Palermo, 59*(2), 199-213. https://doi.org/10.1007/s12215-010-0015-1

7. Wang, L., & Tao, S. (2016). Parameterized Littlewood–Paley operators and their commutators on Herz spaces with variable exponents. *Turkish Journal of Mathematics, 40*(1), 122-145.
8. Karlovich, A. Y., & Lerner, A. K. (2005). Commutators of singular integrals on generalized L p spaces with variable exponent. *Publicacions matematiques,* 111-125.
9. Almeida, A., Hasanov, J., & Samko, S. (2008). Maximal and potential operators in variable exponent Morrey spaces. *Georgian Mathematical Journal, 15*(2), 195-208. https://doi.org/10.1515/GMJ.2008.195
10. Diening, L., Harjulehto, P., Hst, P., & Ruzicka, M. (2011). *Lebesgue and Sobolev spaces with variable exponents.* Springer.
11. Abdalmonem, A., Abalrhman, O., & Tao, S. P. (2016). Commutators of fractional integral with variable kernel on variable exponent Herz and Lebesgue spaces. *International Journal of Mathematics And its Applications, 4,* 29-40.
12. Izuki, M. (2010). Fractional integrals on Herz-Morrey spaces with variable exponent. *Hiroshima Mathematical Journal, 40*(3), 343-355.
13. Izuki, M. (2009). Boundedness of vector-valued sublinear operators on Herz-Morrey spaces with variable exponent. *Mathematical Sciences Research Journal, 13*(10), 243-253.
14. Wang, H. (2016). Commutators of Marcinkiewicz integrals on Herz spaces with variable exponent. *Czechoslovak Mathematical Journal, 66*(1), 251-269. https://doi.org/10.1007/s10587-016-0254-1
15. Lu, Y., & Zhu, Y. P. (2014). Boundedness of multilinear Caldern-Zygmund singular operators on Morrey-Herz spaces with variable exponents. *Acta Mathematica Sinica, English Series, 30*(7), 1180-1194.. https://doi.org/10.1007/s10114-014-3410-2
16. Almeida, A., & Drihem, D. (2012). Maximal, potential and singular type operators on Herz spaces with variable exponents. *Journal of Mathematical Analysis and Applications, 394*(2), 781-795. https://doi.org/10.1016/j.jmaa.2012.04.043
17. Lu, S., Yang, D., & Hu, G. (2008). *Herz type spaces and their applications.* Science Press. Beijing.
18. Stien E, M. (1970). *Singular Integral and Differentiability Properties of Functions.* Princeton, NJ, USA: Princeton University Press.
19. Wu, H., & Lan, J. (2012). The Boundedness for a Class of Rough Fractional Integral Operators on Variable Exponent Lebesgue Spaces. *Analysis in Theory and Applications, 28*(3), 286-293.
20. Abdalmonem, A., Abdalrhman, O., & Tao, S. (2016). Boundedness of fractional integral with variable kernel and their commutators on variable exponent Herz spaces. *Applied Mathematics, 7*(10), 1165.. http://dx.doi.org/10.4236/am.2016.710104
21. Tan, J., & Zhao, J. (2016). Fractional integrals on variable HardyMorrey spaces. *Acta Mathematica Hungarica, 148*(1), 174-190. https://doi.org/10.1007/s10474-015-0571-0
22. Ho, K. P. (2013). The fractional integral operators on Morrey spaces with variable exponent on unbounded domains. *Math. Inequal. Appl, 16,* 363-373. https://doi.org/10.7153/mia-16-27



**Afif Abdalmonem**

Faculty of Science, University of Dalanj, Dalanj, Sudan.
`e-mail: afeefy86@gmail.com`

**Omer Abdalrhman**

College of Education, Shendi University, Shendi, Sudan.
`e-mail: humoora@gmail.com`

**Shuangping Tao**

Northwest Normal University, Lanzhou, China.
`e-mail: taosp@nwnu.edu.cn`