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Reliability Evaluation of Multi-State Flow Networks Via Map Methods

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Authors' contributions

This work was carried out in collaboration between both authors. Author AMAR envisioned and designed the study, performed the symbolic and numerical analysis, solved the detailed examples, constructed the map solutions, managed literature search and substantially edited the entire manuscript. Author OMA implemented the algorithm, contributed to the analysis, prepared the tables, drew the various figures, wrote the first draft of the manuscript and contributed to literature search. Both authors read and approved the final manuscript.

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ABSTRACT

This paper examines two simple (albeit useful) methods used to evaluate the reliability of two-terminal multistate flow networks. These two methods involve two Karnaugh map versions, namely the Variable-Entered Karnaugh Map (VEKM) and the Multi-Valued Karnaugh Map (MVKM). These two versions are crucial in providing not only the visual insight necessary to write better future software but also adequate means of verifying such software. We assess these two versions of map methods versus the exhaustive search method, which guarantees conceptual clarity at the expense of lack of computational efficiency. Our target is the evaluation of the probability mass function (pmf) in a wide array of cases, in which we consider flow from a source node to a sink node in a capacitated network with a multistate capacity model for the links. Each network link has a varying capacity, which is assumed to exist in a mutually exclusive sense. The reliability of the system is wholly dependent on its ability to successfully transmit at least a certain required system flow from the source (transmitter) to the sink (receiver) station. The max-flow min-cut theorem is critical in obtaining all successful states. To demonstrate the proposed methods applicability, two demonstrative examples are given with ample details.

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Keywords: Reliability analysis; multi-state system; flow network; multi-valued Karnaugh map; variable-entered Karnaugh map; max-flow min-cut theorem; exhaustive search method.

1. INTRODUCTION

The existing literature has significantly explored various methods for evaluating the reliability of two-state flow networks [1-5], which are networks of two-state elements. To effectively model a two-state element (whether it is a node or a branch), such element must be assigned only a single fixed capacity, which expresses the maximum flow allowed by that element. According to the two-state model, a node or a branch that cannot transmit the maximum flow (albeit it might transmit a significant amount of flow, i.e. a pronounced portion of the maximum flow), is nevertheless said to be in a failed state. Therefore, there is an obvious need for multistate modelling that is to be achieved through the introduction of one or several intermediate state(s) and assigning each state specific capacities less than the maximum in order to appropriately mimic the practical situation. Undoubtedly, in a multistate model, there are many states for a network element, with each state having an assigned capacity that is less than that element's maximum capacity. This scenario, in a telecommunication network, is depicted by the exchanges (represented as nodes in a reliability graph), which can either be partially or completely healthy. The event of a network exchange being partially healthy occurs when some selectors (in a 20 out of 1000 line exchange, say) are down, which results in the reduction of the exchange's capacity. Hence, according to the multi-state model, an exchange may not be considered to be in a failed state if it has a capacity that is less than the maximum. In a telecommunication network, it must be noted that all the inter-toll trunks or toll-connecting trunks (represented as branches in a reliability graph), may not be healthy all the time, even if they are partially serving the purpose. A prominent related problem is the two-terminal multistate reliability evaluation, which is now considered as a classical network reliability problem. Several methods have been developed that focus on this problem as well as provide critical insights into multistate modelling in general [6-15].

This paper examines two methods of evaluating the reliability of multistate flow networks. The first method involves a map procedure, in which the reliability is evaluated through a Karnaugh map of selective states from capacity vectors. The map constitutes a very powerful manual and simple tool that provides pictorial insight about the various functional properties and procedures pertaining not only to Boolean functions but also to pseudo-Boolean functions [1]. We propose herein the utilization of two variants of the Karnaugh map, namely (a) the Multi-Valued Karnaugh Map (MVKP) [14,16,17], which allows a convenient representation of non-binary discrete random functions, and (b) the Variable-Entered Karnaugh Map (VEKM), which has proven to be an effective tool for increasing the variable-handling capacity of the map [18-21]. Different researchers have coined different names for these variants including basic maps with eliminated variables [22], truth tables with distributed simplifications [23], K-maps within K-maps or reduced Karnaugh maps [24,25] or variable-entered maps [26]. The VEKM has been used successfully in several recent research works as in [27-33].

The second method involves Exhaustive Search (ES) which is also known as brute force, direct search or the generate-and-test method. In spite of the existence of elegant algorithms for evaluating certain multistate flow networks [34,35], exhaustive search still acquires significant acceptability and utility due to the lack of regular structure in many other real-life systems, a limitation that leaves it occasionally as the only possible solution [36-39]. It is worth stating that the exhaustive search (albeit its extreme inefficiency) is sometimes the only method that can systematically enumerate all possible cases for the multistate st network, and check on whether each case can satisfy the problem statement. The complexity of this method deteriorates dramatically with the increase in the size of a capacitated multi state st network. In fact, there is an exponential increase in the number of cases to be considered, which results in infeasible and often prohibitively lengthy searches. However, the recent increase in memory sizes and computing powers, and the emergence of many-core processors as well as the development of parallel and distributed programming has made the utilization of exhaustive search in many problems both feasible and robust, besides reviving interest in brute-force techniques.

The methods proposed herein will not only explore the cutsets of the studied network but will also examine the required flow capacity and the corresponding capacity vectors right from the source node to the sink node. We aligned our discussion of these methods to the two problems explicated by the authors in [34,35].

The remaining part of this paper is organized as follows: Section 2 presents the underlying assumptions for our model and the notation used. Section 3 clarifies the two proposed methods. While the first and main method in this paper depends on Karnaugh maps, the other method uses a MATLAB code for implementing the exhaustive search and determining full characterizations for the pertinent problems, including the probability mass function (pmf) and the complementary cumulative distribution function (CCDF) of network flow, treated as a discrete random variable. Section 4 provides two examples and their solutions are presented to illustrate the proposed techniques for evaluating the reliability of flow network considered as a multi-state system. Section 5 concludes the paper.

2. ASSUMPTIONS AND NOTATION

Assumptions:

The proposed methods are based on the following assumptions.

(1) The physical network is modelled by a linear graph.
(2) Each network branch (edge or link) has a multistate representation, i.e., a representation with a failed state with zero capacity, a state of perfect success with maximum capacity and other states of partial success with capacities less than the maximum capacity.
(3) The probability density function of the capacity for each branch (edge or link) is known.

Notation:

G : Connected graph of flow network $G=(V,B)$
V : A set of nodes (vertices); $v_i \in V$
B : A set of branches (edges or links); $b_i \in B$
s,t : Source and terminal nodes
n_j : The number of states of the jth link
K_l : Be the lth cutset of the flow network, $l=1,2,\ldots,m$
$C(K_l)$: The flow capacity of the lth cut of the flow network, $l=1,2,\ldots,m$
C_s : Required system capacity
C_j^i : Capacity value for the jth branch of the network in the ith state of the capacity vector
C_j : Capacity vector for the jth branch, e.g. $C_1 \equiv (C_1^1, C_1^2, \ldots, C_1^{n1})$
P_j : Probability vector of the capacity vector C_j
R : s-t reliability of the flow network.

3. PROPOSED METHODS

The methods proceed to evaluate the reliability of the multistate flow network, based on knowledge of the following:

(i) The required system capacity.
(ii) The number of states of each element (whether it is a node or a branch).
(iii) Capacities of all states of each branch.
(iv) Probabilities of assigned units of flow for all states of each branch.

We solve each example by two methods. First, we evaluate the given multistate flow network by using two Karnaugh map variants, namely, the Multi-Valued Karnaugh Map (MVKM) that appears in Figs. 2 and 7, which are 4-variable multi-valued maps, and the Variable-Entered Karnaugh Map (VEKM) that appears in Figs. 3 and 8. Our solution strategy follows a divide-and-conquer paradigm. We identify the two most prominent cutsets (involving the smallest numbers of links) of the flow network using any of the methods for minimal cut enumeration [40-42]. Let these two cutsets be denoted by K_l , where $l=1,2$. From the max-flow min-cut theorem [43-45], a flow of the value C_s is attainable in $\boldsymbol{G}$, only where $\min[C(K_l)] \geq C_s$, $l=1,2$, i.e. $[C(K_1) \geq C_s, C(K_2) \geq C_s]$, where $C(K_l)$ is the flow capacity of the cutset K_l. First, all of the possible states of the pertinent branches (constituting the two most prominent cutsets K_l , $l=1,2)$ are identified as cells in a Multi-Valued Karnaugh Map (MVKM), with the corresponding total capacity $C(K_1)+C(K_2)$ being entered in these cells. The states of the pertinent branches which allow the desired amount of flow from the two most prominent cutsets are sorted out. Each of these success states (with a known probability) is considered individually to identify success sub-states within them as specific values of the remaining variables.

The second method considered herein is a computer-based method intended mainly as a verification means for the earlier manual method. It involves exhaustive search, which works by calculating the multi-state truth table for all combinations of branch values by using a specific MATLAB code, and consequently determining the probability mass function (pmf) of the required system capacity C_s as a discrete random variable. The code also characterizes C_s by its complementary cumulative distribution function (CCDF). Either of these two characterizations is a complete characterization of C_s, and can be used to specify the probability of success of the network in maintaining a specific flow value.

4. ILLUSTRATION

Example A:

Fig. 1 shows the Graph Network of Patra and Misra [35], a capacitated multi-state st network, with seven different branches $X_1, X_2, X_3, X_4, X_5, X_6$ and X_7 , with different capacities and probabilities as shown in Table 1.

So, we have $3^4 * 2^3 = 648$ states. And there are 6 minimal cutsets with cutset capacities:

$$C_1 + C_6\ ,\ C_3 + C_7\ , C_2 + C_4 + C_6\ , C_2 + C_5 + C_7\ , C_1 + C_4 + C_5 + C_7, \text{and}\ \ C_3 + C_4 +\ C_5 + C_6$$

The required system capacity in Patra and Misra [35] is $C_s = 6\,u\ nits$ So, we look at the first two cutsets (involving the smallest numbers of links) of the flow network which are of cutset capacities: $C_1 + C_6$, $C_3 + C_7$ and find all possible solutions for $C_1 + C_6 \geq 6$, $C_3 + C_7 \geq 6$. Fig. 2 provides a quick aid for obtaining such solutions. This figure displays a Karnaugh map with map entries representing the combined capacity of the two most prominent cutsets, i.e., the minimum among the capacities of these two cutsets, viz. $F = \min\ (C_1 + C_6, C_3 + C_7)$, as a function of the pertinent branch capacities, which serve as map variables.

As can be seen above from the 4-variable multi-valued-Karnaugh Map, we can simply find the following solutions

(i) The possible vectors corresponding to the requirement $C_1 + C_6 \geq 6$ are $C_1 = 4\ \text{with probability}\ (0.6)\ and\ C_6 = \{2, 3\}\ \text{with probabilities}\ (0.3, 0.5)$.
(ii) The possible vectors corresponding to the requirement $C_3 + C_7\ \geq 6$ are $C_3 = 3\ \text{with probability}\ (0.4)\ and\ \ C_7 = \{3, 4\}\ \text{with probabilities}(0.2, 0.6)$.

Now, our solutions above specify a single value of capacity for branches 1 and 3. Therefore, the original problem splits into subproblems each constituting a 5-branch multi-state st network. Since we demand one of two capacities for each of the branches 6 and 7, we need to study four subproblems, each pertaining to specific capacities for branches 6 and 7. Fig. 3 demonstrates a VEKM for whose four entries are simply the flow capacity for each of these subproblems. This VEKM uses the two variables C_6 and C_7 as map variables, while the remaining five variables are supposedly entered variables. Since two of these have a specific value of capacity (C_1 and C_3), we end up with only three variables C_2, C_4 and C_5 as entered variables.

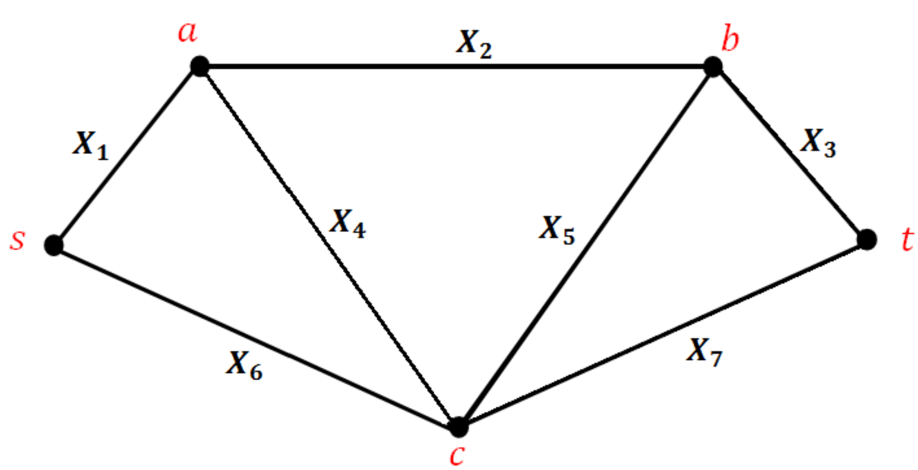

Fig. 1. A capacitated 7-branch multi-state st network with six minimal cutsets

Table 1. Branch capacity vectors and associated probability vectors for the network in Fig. 1

	X_1			X_2		X_3			X_4		X_5		X_6			X_7		
Capacity vectors	0	2	4	0	3	0	1	3	0	4	0	3	0	2	3	0	3	4
Probability vectors	0.1	0.3	0.6	0.2	0.8	0.3	0.3	0.4	0.3	0.7	0.3	0.7	0.2	0.3	0.5	0.2	0.2	0.6

C_3 \ C_1	0 (0.1)			2 (0.3)			4 (0.6)			C_7
0 (0.3)	0	0	0	0	0	0	0	0	0	0 (0.2)
	0	2	3	2	3	3	3	3	3	3 (0.2)
	0	2	3	2	4	4	4	4	4	4 (0.6)
1 (0.3)	0	1	1	1	1	1	1	1	1	0 (0.2)
	0	2	3	2	4	4	4	4	4	3 (0.2)
	0	2	3	2	4	5	4	5	5	4 (0.6)
3 (0.4)	0	2	3	2	3	3	3	3	3	0 (0.2)
	0	2	3	2	4	5	4	6 (0.0144)	6 (0.024)	3 (0.2)
	0	2	3	2	4	5	4	6 (0.0432)	7 (0.072)	4 (0.6)
C_6	0 (0.2)	2 (0.3)	3 (0.5)	0 (0.2)	2 (0.3)	3 (0.5)	0 (0.2)	2 (0.3)	3 (0.5)	

Fig. 2. A 4-variable multi-valued Karnaugh map representation of the combined capacity of the two prominent cutsets (with probabilities of successful states shown in parentheses)

C_7 \ C_6	2 (0.3)	3 (0.5)
3 (0.2)	$\min(6, 6, C_2 + C_4 + 2, C_2 + C_5 + 3, 4 + C_4 + C_5 + 3, 3 + C_4 + C_5 + 2)$ $P_1P_3P_6P_7 = (0.0144)$	$\min(7, 6, C_2 + C_4 + 3, C_2 + C_5 + 3, 4 + C_4 + C_5 + 3, 3 + C_4 + C_5 + 3)$ $P_1P_3P_6P_7 = (0.024)$
4 (0.6)	$\min(6, 7, C_2 + C_4 + 2, C_2 + C_5 + 4, 4 + C_4 + C_5 + 4, 3 + C_4 + C_5 + 2)$ $P_1P_3P_6P_7 = (0.0432)$	$\min(7, 7, C_2 + C_4 + 3, C_2 + C_5 + 4, 4 + C_4 + C_5 + 4, 3 + C_4 + C_5 + 3)$ $P_1P_3P_6P_7 = (0.072)$

Fig. 3. A variable – entered Karnaugh map for the flow capacity $F/C_1\{4\}\ C_3\{3\}$ with map variables C_6 and C_7 (with the probability of each map cell shown parenthetically)

Thus, we have 4 problems that are represented by Karnaugh maps in 3 variables each, namely C_2, C_4 *and* C_5 , which all happen to be binary. Therefore, each of these maps has 8 states only as shown in Figs. 4a-4d. The map entries are the real numbers which represent the flow capacities of the network according to the max-flow min-cut theorem, viz.

$$F = \min(\ C_1 + C_6\ ,\ C_3 + C_7\ , C_2 + C_4 + C_6\ , C_2 + C_5 + C_7\ , C_1 + C_4 + C_5 + C_7,\ \ C_3 + C_4 + C_5 + C_6\ )$$

First case: $C_1 = 4\ (0.6)\ ,\ \ C_3 = 3(0.4)\ ,\ \ C_6 = 2\ (0.3)\ ,\ \ C_7 = 3\ (0.2\ )$

C_5 \ C_2	0 (0.2)		3 (0.8)	
0 (0.3)	$\min(6,6,2,3,7,5) = 2$ $P = 0.0002592$	$\min(6,6,6,3,11,9) = 3$ $P = 0.0006048$	$\min(6,6,9,6,11,9) = 6$ $P = 0.0024192$	$\min(6,6,5,6,7,5) = 5$ $P = 0.0010368$
3 (0.7)	$\min(6,6,2,6,10,8) = 2$ $P = 0.0006048$	$\min(6,6,6,6,14,12) = 6$ $P = 0.0014112$	$\min(6,6,9,9,14,12) = 6$ $P = 0.0056448$	$\min(6,6,5,9,10,8) = 5$ $P = 0.0024192$
C_4	0 (0.3)	4 (0.7)		0 (0.3)

Fig. 4a. Karnaugh Map Representation of the flow capacity $F/C_1\{4\}\ C_3\{3\}\ C_6\{2\}\ C_7\{3\}$ with map variables C_2, C_4 , and C_5

Second case: $C_1 = 4\ (0.6)\ ,\ \ C_3 = 3(0.4),\ \ C_6 = 3\ (0.5)\ ,\ \ C_7 = 3\ (0.2\ )$

C_5 \ C_2	0 (0.2)		3 (0.8)	
0 (0.3)	$\min(7,6,3,3,7,6) = 3$ $P = 0.000432$	$\min(7,6,7,3,11,10) = 3$ $P = 0.001008$	$\min(7,6,10,6,11,10) = 6$ $P = 0.004032$	$\min(7,6,6,6,7,6) = 6$ $P = 0.001728$
3 (0.7)	$\min(7,6,3,6,10,9) = 3$ $P = 0.001008$	$\min(7,6,7,6,14,13) = 6$ $P = 0.002352$	$\min(7,6,10,9,14,13) = 6$ $P = 0.009408$	$\min(7,6,6,9,10,9) = 6$ $P = 0.004032$
C_4	0 (0.3)	4 (0.7)		0 (0.3)

Fig. 4b. Karnaugh Map Representation of the flow capacity $F/C_1\{4\}\ C_3\{3\}\ C_6\{3\}\ C_7\{3\}$ with map variables C_2, C_4 , *and* C_5

Third case: $C_1 = 4\ (0.6)\ ,\ \ C_3 = 3(0.4),\ \ C_6 = 2\ (0.3)\ ,\ \ C_7 = 4\ (0.6\ )$

C_2 / C_5	0 (0.2)		3 (0.8)	
0 (0.3)	$\min(6,7,2,4,8,5) = 2$ $P = 0.0007776$	$\min(6,7,6,4,12,9) = 4$ $P = 0.0018144$	$\min(6,7,9,7,12,9) = 6$ $P = 0.0072576$	$\min(6,7,5,7,8,5) = 5$ $P = 0.0031104$
3 (0.7)	$\min(6,7,2,7,11,8) = 2$ $P = 0.0018144$	$\min(6,7,6,7,15,12) = 6$ $P = 0.0042336$	$\min(6,7,9,10,15,12) = 6$ $P = 0.0169344$	$\min(6,7,5,10,11,8) = 5$ $P = 0.0072576$
C_4	0 (0.3)	4 (0.7)		0 (0.3)

Fig. 4c. Karnaugh Map Representation of the flow capacity $F/C_1\{4\}\ C_3\{3\}\ C_6\{2\}\ C_7\{4\}$ with map variables C_2, C_4, and C_5

Fourth case: $C_1 = 4\ (0.6)$, $C_3 = 3(0.4)$, $C_6 = 3\ (0.5)$, $C_7 = 4\ (0.6)$

C_2 / C_5	0 (0.2)		3 (0.8)	
0 (0.3)	$\min(7,7,3,4,8,6) = 3$ $P = 0.001296$	$\min(7,7,7,4,12,10) = 4$ $P = 0.003024$	$\min(7,7,10,7,12,10) = 7$ $P = 0.012096$	$\min(7,7,6,7,8,6) = 6$ $P = 0.005184$
3 (0.7)	$\min(7,7,3,7,11,9) = 3$ $P = 0.003024$	$\min(7,7,7,7,15,13) = 7$ $P = 0.007056$	$\min(7,7,10,10,15,13) = 7$ $P = 0.028224$	$\min(7,7,6,10,11,9) = 6$ $P = 0.012096$
C_4	0 (0.3)	4 (0.7)		0 (0.3)

Fig. 4d. Karnaugh Map Representation of the flow capacity $F/C_1\{4\}\ C_3\{3\}\ C_6\{3\}\ C_7\{4\}$ with map variables C_2, C_4, *and* C_5

To facilitate probability calculation, we highlight success states (those of capacity not less than 6) in Figs. 4a-4d in yellow, and write the probability of each state within its cell. Hence, the final system reliability is given by the sum of the probabilities of the success states, namely:

$R = \Pr\{\boldsymbol{C_s} \geq 6\} = 0.0024192 + 0.0014112 + 0.0056448 + 0.004032 + 0.001728 + 0.002352 + 0.009408 + 0.004032 + 0.0072576 + 0.0042336 + 0.0169344 + 0.012096 + 0.005184 + 0.007056 + 0.028224 + 0.012096 = \mathbf{0.1241088}$, which can be seen as a similar result to the work of Patra and Misra [35].

The second method to be considered herein is that of exhaustive search. This method calculates the truth table for all possible combinations implied by Table 1. The number of these combinations (also called cases or states) is $3^4 * 2^3 = 648$ cases. We constructed a special MATLAB code to calculate the maximum possible flow for each of these deterministic cases, using the max-flow min-cut theorem, namely. Flow in a particular deterministic state = min {cutset capacity} = $\min[C(k_l)]$, where $C(k_l)$ is the flow capacity of the cutset K_l, which is the lth cutset of the network, and the minimization operator (min) is performed over $l = 1,2,\ldots 6$. Table 2 displays numerical values of the probability mass function (pmf) and the complementary cumulative distribution function (CCDF) of the random flow variable. The results in this table match (within the accuracy used) those obtained with our former map procedure at the required system capacity of $C_s = 6\ units$. Though the exhaustive search method is more computationally demanding, it is more informative as it gives a result at any required system capacity. Figs. 5a and 5b shows the corresponding graphical representations of the probability mass function and the complementary cumulative distribution function of the random flow variable. In passing, we note that this discrete random flow variable would have still been multi-valued, even if all the input variables had been binary (see, e.g., [1]). The increase in difficulty in going from a flow network with binary components to one with multi-state components does not pertain to the form of the final solution, but is implicit in the approach needed to achieve this solution.

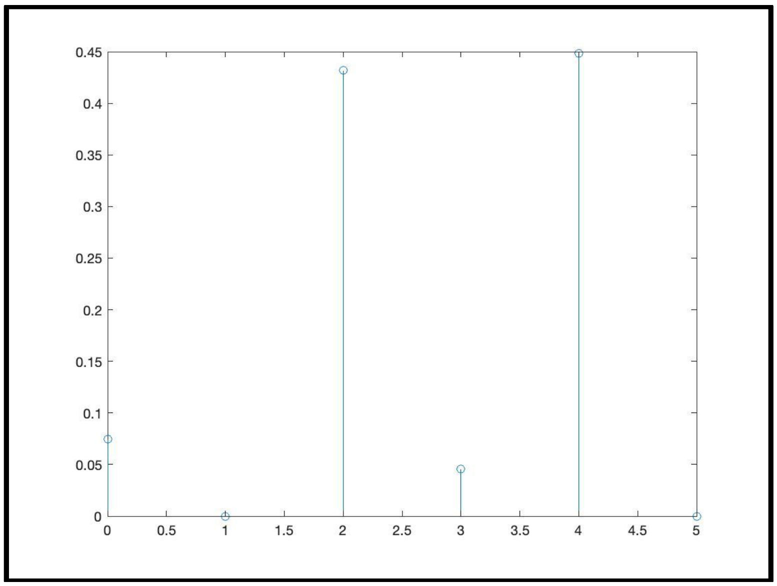

Fig. 5a. A plot of the pmf with all required cases

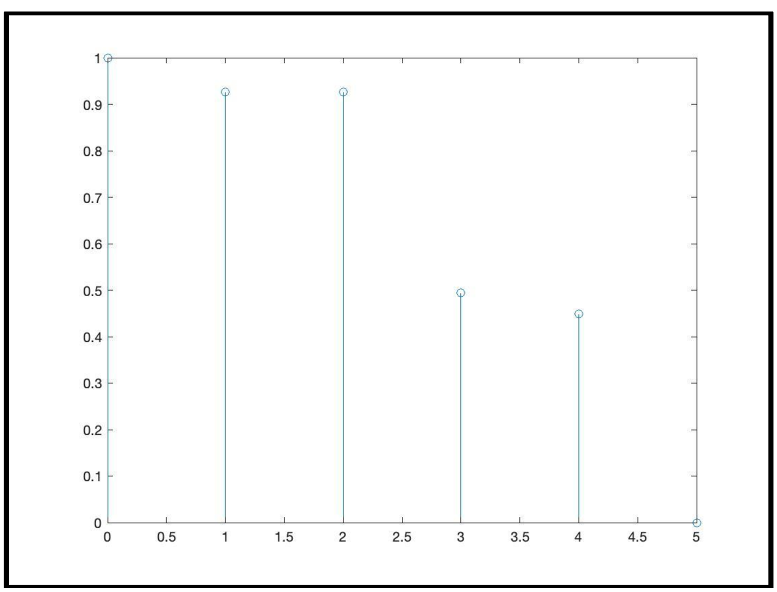

Fig. 5b. A plot of the CCDF with all required cases

Table 2. The probability mass function (pmf) and the complementary cumulative distribution function for the random flow variable

Flow	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
pmf	0.10065	0.057583	0.093355	0.20436	0.26935	0.1506	0.076733	0.047376	0.00
Case	F > -1	F > 0	F > 1	F > 2	F > 3	F > 4	F > 5	F > 6	F > 7
CCDF	1.00	0.8993	0.8418	0.7484	0.5441	0.2747	0.1241	0.0474	0.00

Example B:

Fig. 6 shows the Graph Network of Sharma, et al. [34], a capacitated multi-state st network, with five different branches X_1, X_2, X_3, X_4 and X_5 with different capacities and probabilities as shown in Table 3.

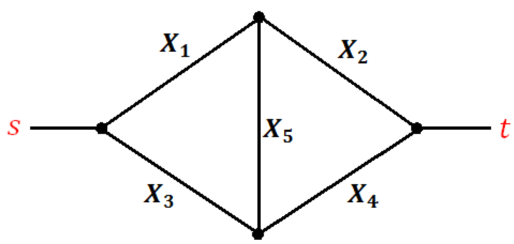

Fig. 6. A capacitated 5-branch multi-state st network with four minimal cutsets

We have $2 * 2 * 3 * 2 * 2 = 48$ states. And there are 4 minimal cutsets with cutset capacities:

$C_1 + C_3$, $C_2 + C_4$, $C_1 + C_5 + C_4$, and $C_2 + C_5 + C_3$

The required system capacity in Sharma, et al. [34] is $C_s = 4\,units$ So, we look at the first two cutsets (involving the smallest numbers of links) of the flow network which are of cutset capacities: $C_1 + C_3$, $C_2 + C_4$ and find all possible solutions for $C_1 + C_3 \geq 4$, $C_2 + C_4 \geq 4$. Fig. 7 provides a quick aid for obtaining such solutions. This figure displays a Karnaugh map with map entries representing the combined capacity of the two most prominent cutsets, i.e., the minimum among the capacities of these two cutsets, viz. $F = \min(C_1 + C_3 , C_2 + C_4)$, as a function of the pertinent branch capacities, which serve as map variables.

Table 3. Branch capacity vectors and associated probability vectors for the network in Fig. 6

	X_1		X_2		X_3			X_4		X_5	
Capacity vectors	0	2	0	2	0	3	4	0	2	0	4
Probability vectors	0.2	0.8	0.1	0.9	0.2	0.4	0.4	0.3	0.7	0.1	0.9

C_1 \ C_2	0 (0.2)			2 (0.8)			C_4
0 (0.1)	0	0	0	0	0	0	0 (0.3)
	0	2	2	2	2	2	2 (0.7)
2 (0.9)	0	2	2	2	2	2	0 (0.3
	0	3	4 (0.0504)	2	4 (0.2016)	4 (0.2016)	2 (0.7)
C_3	0 (0.2)	3 (0.4)	4 (0.4)	0 (0.2	3 (0.4)	4 (0.4)	

Fig. 7. A 4-variable multi-valued Karnaugh map representation of the combined capacity of the two prominent cutsets (with probabilities of successful states shown in parentheses)

As can be seen above from the 4-variable multi-valued-Karnaugh Map, we can simply find the following solutions

(i) The possible vectors corresponding to the requirement $C_1 + C_3 \geq 4$ are $C_1 = \{0,2\}$ with probability $(0.2,0.8)$ and $C_3 = \{3,4\}$ with probabilities $(0.4, 0.4)$.
(ii) The possible vectors corresponding to the requirement $C_2 + C_4 \geq 4$ are $C_2 = 2$ with probability (0.9) and $C_4 = 2$ with probabilities (0.7).

Now, our solutions above specify a single value of capacity for branches 2 and 4. Therefore, the original problem splits into subproblems each constituting a 3-branches multi-state st network. Since we demand one of two capacities for each of the branches 1 and 3, we need to study four subproblems, each pertaining to specific capacities for branches 1 and 3. Fig. 8 demonstrates a VEKM for whose four entries are simply the flow capacity for each of these subproblems. This VEKM uses the two variables C_1 and C_3 as map variables, while the remaining three variables are supposedly entered variables. Since two of these have a specific value of capacity (C_2 and C_4), we end up with only one variable C_5 as an entered variable.

C_3 \ C_1	0 (0.2)	2 (0.8)
4 (0.4)	$min(4, 4, 2 + C_5, 6 + C_5)$ $P_1P_2P_3P_4 = (0.0504)$	$\min(6, 4, 4 + C_5, 6 + C_5)$ $P_1P_2P_3P_4 = (0.2016)$
3 (0.4)	$min(3, 4, 2 + C_5, 5 + C_5)$ $P_1P_2P_3P_4 = (0.0504)$	$min(5, 4, 4 + C_5, 5 + C_5)$ $P_1P_2P_3P_4 = (0.2016)$

Fig. 8. A variable – entered Karnaugh map for the flow capacity $F/C_2\{2\}\ C_4\{2\}$ with map variables C_1 and C_3(with the probability of each map cell shown parenthetically)

Thus, we have 4 problems that are represented by Karnaugh maps in one variable each, namely C_5, which all happens to be a binary variable. Therefore, each of these maps has 2 states only as shown in Figs. 9a-9d. the map entries are the real numbers which represent the flow capacities of the network according to the max-flow min-cut theorem, viz.

$$F = \min(C_1 + C_3, C_2 + C_4, C_1 + C_5 + C_4, C_2 + C_5 + C_3)$$

First state: $C_1 = 0\ (0.2)$, $C_2 = 2(0.9)$, $C_3 = 4\ (0.4)$, $C_4 = 2\ (0.7)$

C_5	0 (0.1)	4 (0.9)
	$\min(4,4,2,6) = 2$ $P = 0.00504$	$\min(4,4,6,10) = 4$ $P = 0.04536$

Fig. 9a. Karnaugh Map Representation of the flow capacity $F/C_1\{0\}C_2\{2\}C_3\{4\}C_4\{2\}$ with map variable C_5

Second state: $C_1 = 2\ (0.8)$, $C_2 = 2(0.9)$, $C_3 = 4\ (0.4)$, $C_4 = 2\ (0.7)$

C_5	0 (0.1)	4 (0.9)
	$\min(6,4,4,6) = 4$ $P = 0.02016$	$\min(6,4,8,10) = 4$ $P = 0.18144$

Fig. 9b. Karnaugh Map Representation of the flow capacity $F/C_1\{2\}C_2\{2\}C_3\{4\}C_4\{2\}$ with map variable C_5

Third state: $C_1 = 0\ (0.2)$, $C_2 = 2(0.9)$, $C_3 = 3\ (0.4)$, $C_4 = 2\ (0.7\ )$

C_5	0 (0.1)	4 (0.9)
	$\min(3,4,2,5) = 2$ P =0.00504	$\min(3,4,6,9) = 3$ P =0.04536

Fig. 9c. Karnaugh Map Representation of the flow capacity $F/C_1\{0\}C_2\{2\}C_3\{3\}C_4\{2\}$ with map variable C_5

Fourth state: $C_1 = 2\ (0.8)$, $C_2 = 2(0.9)$, $C_3 = 3\ (0.4)$, $C_4 = 2\ (0.7\ )$

C_5	0 (0.1)	4 (0.9)
	$\min(5,4,4,5) = 4$ $P = 0.02016$	$\min(5,4,8,9) = 4$ $P = 0.18144$

Fig. 9d. Karnaugh Map Representation of the flow capacity $F/C_1\{2\}C_2\{2\}C_3\{3\}C_4\{2\}$ with map variable C_5

To facilitate probability calculation, we highlight success states (those of capacity not less than 4) in Figs. 9a-9d in yellow and write the probability of each state within its cell. Hence, the final system reliability is given by the sum of the probabilities of the success states, namely:

$R = \Pr\{\boldsymbol{C_s} \geq 4\} = 0.04536 + 0.02016 + 0.18144 + 0.02016 + 0.18144 = \mathbf{0.44856}$, which can be seen as a similar result to the work of Sharma, et al. [34].

The second method to be considered herein is that of exhaustive search. This method calculates the truth table for all possible combinations implied by Table 3. The number of these combinations (also called cases or states) is $3^1 * 2^4 = 48$ cases. We constructed a special MATLAB code to calculate the maximum possible flow for each of these deterministic cases, using the max-flow min-cut theorem, namely. Flow in a particular deterministic state = min {cutset capacity} = $\min[C(k_l)]$,where $C(k_l)$ is the flow capacity of the cutset K_l, which is the lth cutset of the network, and the minimization operator (min) is performed over $l = 1,2,3,$ and 4. Table 5 displays numerical values of the probability mass function (pmf) and the complementary cumulative distribution function (CCDF) of the random flow variable. The results in this table match (within the accuracy used) those obtained with our former map procedure at the required system capacity of $C_s = 4\ u\ nits$. Though the exhaustive search method is more computationally demanding, it is more informative as it gives a result at any required system capacity. Figs. 10a and 10b shows the corresponding graphical representations of the probability mass function and the complementary cumulative distribution function of the random flow variable. In contrast to the problem of Example 1, which has a prohibitively large state space of 648 configurations, we are able in our present case to report a complete listing of the pertinent truth table (of 48 lines) in Table 4. Of course, it would have been better to have offered this truth table in Karnaugh map format.

Table 4. Truth table for all 48 cases of example 2

X_1	X_2	X_3	X_4	X_5	Flow	Probability	X_1	X_2	X_3	X_4	X_5	Flow	Probability
0	0	0	0	0	0	0.00012	2	0	0	0	0	0	0.00048
0	0	0	0	4	0	0.00108	2	0	0	0	4	0	0.00432
0	0	0	2	0	0	0.00028	2	0	0	2	0	0	0.00112
0	0	0	2	4	0	0.00252	2	0	0	2	4	2	0.01008
0	0	3	0	0	0	0.00024	2	0	3	0	0	0	0.00096
0	0	3	0	4	0	0.00216	2	0	3	0	4	0	0.00864
0	0	3	2	0	2	0.00056	2	0	3	2	0	2	0.00224
0	0	3	2	4	2	0.00504	2	0	3	2	4	2	0.02016
0	0	4	0	0	0	0.00024	2	0	4	0	0	0	0.00096
0	0	4	0	4	0	0.00216	2	0	4	0	4	0	0.00864
0	0	4	2	0	2	0.00056	2	0	4	2	0	2	0.00224
0	0	4	2	4	2	0.00504	2	0	4	2	4	2	0.02016
0	2	0	0	0	0	0.00108	2	2	0	0	0	2	0.00432
0	2	0	0	4	0	0.00972	2	2	0	0	4	2	0.03888
0	2	0	2	0	0	0.00252	2	2	0	2	0	2	0.01008
0	2	0	2	4	0	0.02268	2	2	0	2	4	2	0.09072
0	2	3	0	0	0	0.00216	2	2	3	0	0	2	0.00864
0	2	3	0	4	2	0.01944	2	2	3	0	4	2	0.07776
0	2	3	2	0	2	0.00504	2	2	3	2	0	4	0.02016
0	2	3	2	4	3	0.04536	2	2	3	2	4	4	0.18144
0	2	4	0	0	0	0.00216	2	2	4	0	0	2	0.00864
0	2	4	0	4	2	0.01944	2	2	4	0	4	2	0.07776
0	2	4	2	0	2	0.00504	2	2	4	2	0	4	0.02016
0	2	4	2	4	4	0.04536	2	2	4	2	4	4	0.18144

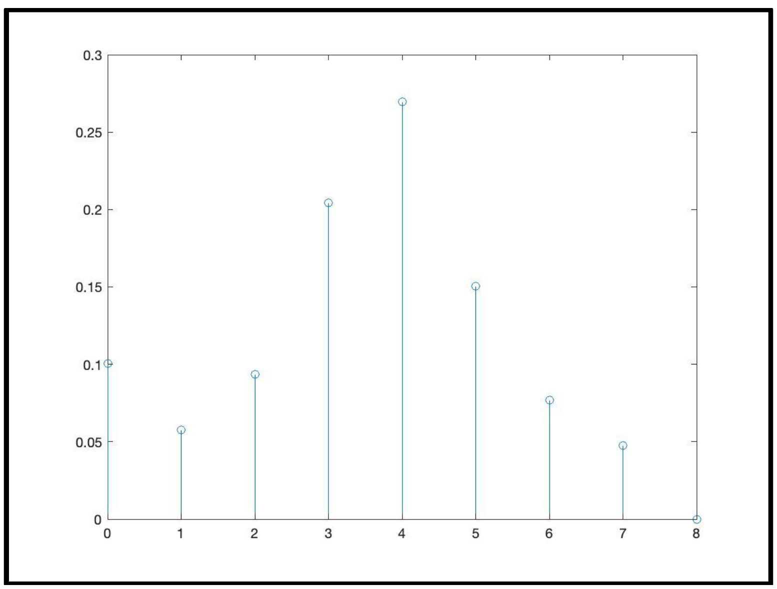

Fig. 10a. A plot of the pmf for Example 2

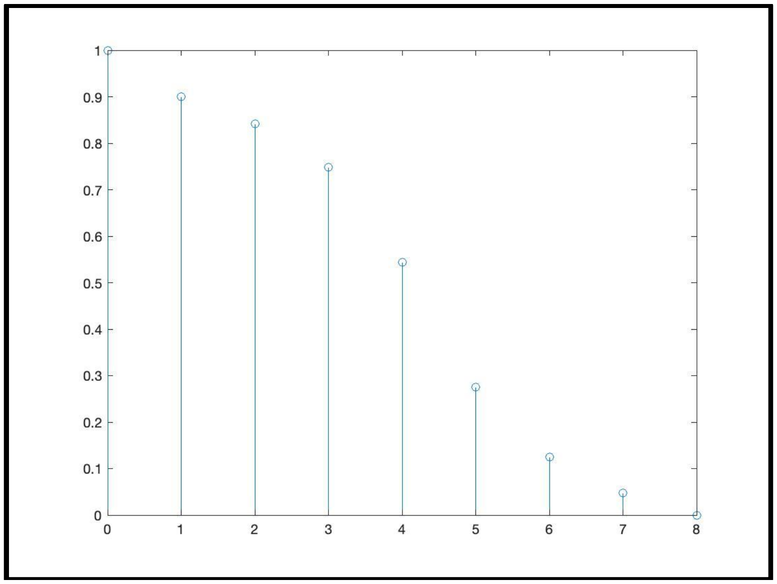

Fig. 10b. A plot of the CCDF for Example 2

Table 5. Probability mass function (pmf) and the complementary cumulative distribution function (CCDF) for Example 2

Flow	**0**	**1**	**2**	**3**	**4**	**5**
pmf	0.07424	0.00	0.43184	0.04536	0.44856	0.00
Case	F > -1	F > 0	F > 1	F > 2	F > 3	F > 4
CCDF	1.00	0.9258	0.9258	0.4939	0.4486	0.00

5. CONCLUSION

This paper presents two simple and efficient methods for the reliability evaluation of a multistate network, in which a specific commodity flows from a source node to a sink node with required system capacity constraints. Both methods rely on repeated application of the max-flow min-cut theorem for each individual deterministic state of the random network. However, while the second method insists on addressing every individual state *per se*, the first method attains some computational efficiency by treating groups of similar states collectively. The first method involves a map procedure, in which the reliability is evaluated through a Karnaugh map of selective states from capacity vectors. This method gives a reliability value at only a certain required system capacity. The advantage of this method that it is a simple and powerful manual approach that provides pictorial insight about the various functional properties and procedures. The second method involves Exhaustive Search (ES) which is also known as an approach of brute force, direct search or generate and test. This method provides more information, at the expense of a more demanding computational cost that is so terribly severe as to make a manual implementation of it almost out of question. Through this method, we determined the pmf and the CCDF of the flow random variable, thereby achieving a complete characterization of this variable, rather than just computing a single probability of its exceeding a specific threshold.

As we stated earlier, our discrete random flow (output) variable would have still been multi-valued, even if all the input variables had been binary. The increase in complexity arising by going from a flow network with binary components to one with multi-state components does not pertain to the form of the final solution, but it is inherent in the approach needed to achieve this solution.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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