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Asymptotic Analysis of the Static and Dynamic Buckling of a Column with Cubic - Quintic Nonlinearity Stressed by a Step Load

A. M. Ette[1], J. U. Chukwuchekwa[1*], I. U. Udo–Akpan[2] and W. I. Osuji[1]

[1]Department of Mathematics, Federal University of Technology, Owerri, Imo State, Nigeria.
[2]Department of Mathematics and Statistics, University of Port Harcourt, Port Harcourt, Rivers State, Nigeria.

Authors' contributions

AME is the head of the research team and he proposed the problem. All authors namely, AME, JUC, IUUA and WIO solved the problem independently. All authors took a unanimous decision on the final draft. Author JUC did the secretarial work, did the numerical computations and graphical plots and served as the corresponding author.

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Abstract

In this paper, the static and dynamic buckling loads of a viscously damped imperfect finite column lying on an elastic foundation with cubic – quintic nonlinearity but trapped by a step load (in the dynamic case) is investigated analytically. The main objective is to determine analytically both the static and dynamic buckling loads by means of perturbation and asymptotic procedures and relate both buckling loads in one single formula. The formulation contains small perturbations particularly in the viscous damping and imperfection amplitude. Multi – timing perturbation techniques and asymptotics are easily utilized in analyzing the problem. The results, which are nontrivially obtained, are implicit in nature and are valid as long as the magnitudes of the small perturbations become asymptotically small compared to unity.

Keywords: Non-linear elastic foundations; dynamic buckling; step load; perturbation and asymptotic analyses.

**Corresponding author: E-mail: joyuchekwa@gmail.com;*

1 Introduction

Investigations concerning static and dynamic buckling of structures under prescribed loading histories, have received tremendous patronage in recent times. Such investigations include studies by Chitra and Priyardarsini [1], Ferri et al. [2], Kolakowski [3], Kowal-Michalska [4]and Mcshane et al. [5]. Priyardarsini et al. [6] investigated numerical and experimental study of advanced fiber composite cylinders under axial compression while Reda and Forbes [7] studied the dynamic effect of lateral buckling of high temperature / high pressure of off shore pipelines. Of special note is the investigation by Belyav et al. [8], who investigated the stability of transverse vibration of rod under longitudinal step – wise loading while Kripka and Martin [9] investigated cold – formed steel channel columns optimization with simulated annealing method. Similarly, Jatav and Datta [10] investigated shape optimization of damaged columns subjected to conservative and non – conservative forces while Artem and Aydin [11] studied exact solution and dynamic buckling analysis of beam – column loading.

Worthy of mention are also the following recent research works on the buckling of elastic structures and the effects of imperfection on the structures. Patil et al. [12] reviewed the buckling analyses of various structures like plates and shells while Hu and Burgue*ñ*o [13] studied the elastic post buckling response of axially – loaded cylindrical shells with seeded geometric imperfection design. In the same way, Ziolkowski and Imieowski [14] discussed the buckling and post buckling behaviour of prismatic aluminium column submitted to a series of compressive loads while Adman and Saidani [15] discussed the elastic buckling of columns with end restraint effects. Similarly, Avcar [16] studied the elastic buckling of steel columns under axial compression while Kriegesmann et al. [17] studied sample size dependent probabilistic design of axially compressed cylindrical shells.

This work is an extension of previous works on columns to the case where the column lies on a cubic-quintic nonlinear elastic foundation. It is to be noted that most other works had dealt with columns on cubic foundations or quadratic – cubic nonlinear foundations. Works on columns with cubic - quintic nonlinearity are rare.

2 Governing Equation of Motion

The governing differential equation satisfied by the normal displacement W(X, T) of the viscously damped column trapped by an arbitrary load P(T) is as follows [18]:

$$mW_{,TT} + c_0 W_{,T} + EIW_{,XXXX} + 2PW_{,XX} + k_1 W + \alpha k_2 W^3 - \beta_1 k_3 W^5 = -2P(T)\frac{d^2\underline{W}}{dX^2}, \quad T > 0, \tag{1}$$

$$0 < X < \pi, \tag{2}$$

$$W(X,0) = W_{,T}(X,0) = 0, \quad 0 < X < \pi, \qquad c_0 > 0 \tag{3}$$

$$W = W_{,XX} = 0 \; at \; X = 0, \pi, \;\; T \geq 0 \tag{4}$$

where, m is the mass per unit length of the finite column, c_0 is the positive but light viscous damping coefficient, EI is the bending stiffness, where E and I are the Young's modulus and the moment of inertia respectively,$\alpha \; and \; \beta_1$ are exponents of imperfection sensitivity parameters which are to be chosen such that the structure is imperfection sensitive, $\underline{W}$ is the stress – free time independent but twice – differentiable imperfection while X and T are spatial coordinate and time variable respectively. The cubic – quintic nonlinear elastic foundation exerts a force per unit length given by $k_1 W + \alpha k_2 W^3 - \beta_1 k_3 W^5$ on the column. All nonlinearities higher than quintic are neglected while all nonlinear derivatives are similarly neglected. Here, a subscript after a comma indicates partial differentiation.

3 Nondimensionalization of the Governing Equations

Let

$$X = \left(\frac{EI}{k_1}\right)^{\frac{1}{4}} x, \quad W = \left(\frac{k_1}{k_2}\right)^{\frac{1}{2}} w, \quad \lambda f(t) = \frac{P(T)}{2(EIk_1)^{\frac{1}{2}}}, \quad \underline{W} = \epsilon \left(\frac{k_1}{k_2}\right)^{\frac{1}{2}} \varpi$$

$$t = \left(\frac{k_1}{m}\right)^{\frac{1}{2}} T, \quad 2\epsilon^2 = \frac{c_0}{(mk_1)^{\frac{1}{2}}}, \quad \beta = \left(\frac{\beta_1 k_1 k_3}{k_2^2}\right)^{\frac{3}{2}}$$

On substituting these non-dimensional quantities into the governing equations, the resultant equations are

$$w_{,tt} + 2\epsilon^2 w_{,t} + w_{,xxxx} + 2\lambda f(t) w_{,xx} + w + \alpha w^3 - \beta w^5 = -2\lambda\epsilon f(t)\frac{d^2\varpi}{dx^2}, \quad t > 0 \tag{5}$$

$$0 < x < \pi \tag{6}$$

$$w(x,0) = w_{,t}(x,0) = 0, \quad 0 < x < \pi \tag{7}$$

$$w = w_{,xx}(x,0) = 0 \; at \; x = 0, \pi, \quad t \geq 0 \tag{8}$$

Here, simply -supported boundary conditions are assumed, while ϵ and λ are small parameters satisfying the inequalities $0 < \epsilon \ll 1$, and $0 < \lambda < 1$. Physically, ϵ denotes the amplitude of imperfection while λ is that of the applied load and $f(t)$ is the actual time dependent load function, which, in this investigation, is the step load given by

$$f(t) = \{1, \quad t > 0 \;\; and \;\; 0 \, for \quad t < 0 \tag{9}$$

4 Solution of the Associated Static Problem

The governing equation in this case is

$$\frac{d^4 w}{dx^4} + 2\lambda \frac{d^2 w}{dx^2} + w + \alpha w^3 - \beta w^5 = -2\lambda\epsilon \frac{d^2\varpi}{dx^2}, \quad 0 < x < \pi \tag{10a}$$

$$w = \frac{d^2 w}{dx^2} = 0 \; at \; x = 0, \pi \tag{10b}$$

Let

$$w = \sum_{i=1}^{\infty} V^{(i)}(x)\epsilon^i \tag{11}$$

By equating coefficients of powers of ϵ, the following equations are easily obtained.

$$O(\epsilon): \frac{d^4 V^{(1)}}{dx^4} + 2\lambda \frac{d^2 V^{(1)}}{dx^2} + V^{(1)} = -2\lambda\epsilon \frac{d^2\varpi}{dx^2} \tag{12}$$

$$O(\epsilon^2): \frac{d^4 V^{(2)}}{dx^4} + 2\lambda \frac{d^2 V^{(2)}}{dx^2} + V^{(2)} = 0 \tag{13}$$

$$O(\epsilon^3):\ \frac{d^4V^{(3)}}{dx^4}+\ 2\lambda\frac{d^2V^{(3)}}{dx^2}+\ V^{(3)}=\ -\alpha\left(V^{(1)}\right)^3 \tag{14}$$

$$O(\epsilon^4):\ \frac{d^4V^{(4)}}{dx^4}+\ 2\lambda\frac{d^2V^{(4)}}{dx^2}+\ V^{(4)}=\ -3\alpha\left(V^{(1)}\right)^2V^{(2)} \tag{15}$$

$$O(\epsilon^5):\ \frac{d^4V^{(5)}}{dx^4}+\ 2\lambda\frac{d^2V^{(5)}}{dx^2}+\ V^{(5)}=\ -3\alpha\left({V^{(1)}}^2V^{(3)}+V^{(1)}{V^{(2)}}^2\right)+\ \beta {V^{(1)}}^5 \tag{16}$$

etc.

Let

$$\varpi=\ \underline{a}_m sin\ mx\ ,\quad V^{(i)}(x)=\ \sum_{n=1}^{\infty}V_n^{(i)}sinnx \tag{17}$$

Substituting Eq. (17) into (12) yields

$$\sum_{n=1}^{\infty}(n^4-2n^2\lambda+1)\,V_n^{(1)}sinnx=2\lambda m^2\underline{a}_m sin\ mx \tag{18}$$

Multiplying (18) by $sin\ mx$, gives

$$V_m^{(1)}=\ \frac{2\lambda m^2\underline{a}_m}{m^4-2m^2\lambda+1}=B \tag{19a}$$

The solution of (13) easily yields

$$V^{(3)}=0 \tag{19b}$$

Equation (14) now takes the form

$$\frac{d^4V^{(3)}}{dx^4}+\ 2\lambda\frac{d^2V^{(3)}}{dx^2}+\ V^{(3)}=\ -\frac{\alpha B^3}{4}(3sin\ mx-sin\ 3mx\ \ ) \tag{20}$$

On substituting (17) in (20), it is observed that, when $n=m$, the result is

$$V_m^{(3)}=\ \frac{-3\alpha B^3}{4\theta^2},\quad \theta^2=\ (m^4-2m^2\lambda+1)>0 \tag{21a}$$

However, for $n=3m$, the result is

$$V_{3m}^{(3)}=\ \frac{-B^3\alpha}{4\theta\omega^2},\quad \omega^2=\ (81m^4-18m^2\lambda+1)>0\ \forall\, m \tag{21}$$

Thus, for $V^{(3)}$, we get

$$V^{(3)}=\ V_m^{(3)}sin\ mx+\ V_{3m}^{(3)}sin\ 3mx \tag{22}$$

On substituting in (15), using (19b), the following result is obtained

$$V^4\ \equiv 0$$

Substitution is next made into (16), using (19b) and (22) to get

$$\frac{d^4V^{(5)}}{dx^4}+2\lambda\frac{d^2V^{(5)}}{dx^2}+V^{(5)}$$
$$=-3\alpha\left[\frac{1}{2}V^{(1)^2}\left(\frac{3V_m^{(3)}}{2}-V_{3m}^{(3)}\right)\sin mx+\frac{1}{2}\left(V_m{}^{(1)^2}V_{3m}^{(3)}-\frac{1}{4}V_m{}^{(1)^2}V_m^{(3)}\right)\sin 3mx\right.$$
$$\left.+\frac{1}{4}V^{(1)^2}V_{3m}^{(3)}\sin 5mx\right]$$
$$+\frac{\beta B^5}{16}(11\sin mx-5\sin 3mx+\sin 5mx) \qquad (23)$$

Using (17) for $n=m$, we get

$$V_m^{(5)}=\left[\frac{-3\alpha}{\theta^2}\left\{\frac{1}{2}V^{(1)^2}\left(\frac{3}{2}V_m^{(3)}-V_{3m}^{(3)}\right)\right\}+\frac{11\beta}{16\theta^2}\right]\sin mx \qquad (24)$$

However, for $n=3m$, the result is

$$V_{3m}^{(5)}=\left[\frac{5\beta B^5}{16\omega^2}-\frac{-3\alpha}{\omega^2}\left\{\frac{1}{2}V_m{}^{(1)^2}V_{3m}^{(3)}-\frac{1}{4}V^{(1)^2}V_{3m}^{(3)}\right\}\right]\sin 3mx \qquad (25a)$$

Now, when $n=5m$ in (23), we get

$$V_{5m}^{(5)}=\frac{1}{\varphi^2}\left[\frac{V_m^{(1)^5}}{16}-\frac{3\alpha V_m^{(1)^2}}{4}V_{3m}^{(3)}\right]\sin 5mx \qquad (25b)$$

$$\varphi^2=(625m^4-50m^2\lambda+1)>0\ \forall m \qquad (25c)$$

It follows that

$$V^{(5)}=V_m^{(5)}\sin mx+V_{3m}^{(5)}\sin 3mx+V_{5m}^{(5)}\sin 5mx \qquad (26)$$

Thus, the displacement at static loading is

$$w(x)=\epsilon V_m^{(1)}\sin mx+\epsilon^3\left(V_m^{(3)}\sin mx+V_{3m}^{(3)}\sin 3mx\right)$$
$$+\epsilon^3\left(V_m^{(5)}\sin mx+V_{3m}^{(5)}\sin 3mx+V_{3m}^{(5)}\sin 5mx\right)+\cdots \qquad (27)$$

5 Static Buckling Load

As in Amazigo [18], the static buckling load λ_S is obtained from the maximization

$$\frac{d\lambda}{dw}=0 \qquad (28)$$

It should be noted that $V_m^{(i)}$ depends on the load parameter λ through B. The static buckling load will here be given in two separate approximations, first, by taking the displacement strictly in the shape of the imperfection and next, by admitting the buckling mode in the combined shape of sin 3mx alongside the mode in the shape of imperfection.

5.1 Static Buckling Load Using Modes in the Shape of Imperfection

Here,

$$w(x)=\left(\epsilon V_m^{(1)}+\epsilon^3V_m^{(3)}+\epsilon^5V_m^{(5)}\right)\sin mx+\cdots \qquad (29)$$

where,

$$V_m^{(1)} = B, \qquad V_m^{(3)} = \frac{-3\alpha B^3}{4\theta^2}, \qquad V_m^{(5)} = \frac{B^5 Q_{46}}{\theta^2} \tag{30}$$

$$Q_{46} = \frac{11\beta}{16} - 3\alpha^2\left\{\frac{1}{8}\left(\frac{9}{2\theta^2} + \frac{1}{\omega^2}\right)\right\} \tag{31}$$

Equation (29) shall be determined at a convenient point, namely at $x = \frac{\pi}{2m}$, where $\frac{dw}{dx} = 0$. This gives

$$w(x) = \epsilon V_m^{(1)} + \epsilon^3 V_m^{(3)} + \epsilon^5 V_m^{(5)} + \cdots \tag{32}$$

Let $V_m^{(1)} = c_1, \qquad V_m^{(3)} = c_2, \qquad V_m^{(3)} = c_3.$

Consequently, the result becomes

$$w = \epsilon c_1 + \epsilon^3 c_2 + \epsilon^5 c_3 + \cdots \tag{33}$$

It should be noted that the choice of $x = \frac{\pi}{2m}$ follows from the fact that the accompanying dynamic problem will eventually be determined at the same point. As in Amazigo [18], the series (33) does not converge when $w > w_a$, where w_a is the displacement at buckling. The difficulty is overcome by reversing (33) in the form

$$\epsilon = d_1 w + d_2 w^3 + d_3 w^5 + \cdots \tag{34}$$

By substituting for w from (33) in (34) and equating the coefficients of powers of ϵ, the following are obtained

$$d_1 = \frac{1}{c_1}, \qquad d_2 = \frac{-c_2}{c_1^4}, \qquad d_3 = \frac{3c_2^2 - c_1 c_3}{c_1^7} \tag{35a}$$

i.e. $\quad d_1 = \frac{1}{B}, \qquad d_2 = \frac{-3\alpha}{4B\theta^2}, \qquad d_3 = \frac{-11Q_{46}}{16B}\left(\frac{\beta}{\theta^2}\right)Q_{47}$

$$Q_{47} = \left[\frac{27}{11Q_{46}}\left(\frac{\alpha^2}{\beta}\right) - 1\right] \tag{35b}$$

The maximization (28) is now accomplished through (34) to yield

$$d_1 + 3d_2 w_a^2 + 5d_3 w_a^4 = 0 \tag{36}$$

This yields

$$w_a^2 = \frac{-3d_2 \pm \sqrt{9d_2^2 - 20d_1 d_3}}{10d_3} \tag{37}$$

By taking the positive square root sign, equation (37) becomes

$$w_a^2 = \frac{18\alpha}{55\beta Q_{46} Q_{47}}\left[\sqrt{1 + \frac{220 Q_{46} Q_{47}}{99}} - 1\right] \tag{38}$$

Therefore, the following result is finally obtained

$$w_a = \frac{3\sqrt{\frac{2}{55}}\left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}}}{\sqrt{Q_{46} Q_{47}}}\left[\sqrt{1 + \frac{220 Q_{46} Q_{47}}{99}} - 1\right]^{\frac{1}{2}} \tag{39}$$

where, (38) and (39) are determined at $\lambda = \lambda_S$. The static buckling load in the case of buckling modes strictly in the shape of imperfection is determined by evaluating (34) at static buckling condition, where $w = w_a$ and$\lambda = \lambda_S$.This gives

$$\epsilon = w_a[d_1 + d_2 w_a^2 + d_3 w_a^4] \tag{40}$$

Multiplying (40) by 5 and after, substituting for $5d_3 w_a^4$ from (36) and simplifying, the following is obtained

$$5\epsilon = \frac{4w_a(m^4 - 2m^2\lambda_S + 1)}{m^2\underline{a}_m\lambda_S}\left[1 - \frac{3\alpha w_a^2(\lambda_S)}{8\theta^2}\right] \tag{41}$$

Here, the static buckling load λ_S is given implicitly through B, Q_{46} and Q_{47}.

5.2 Static Buckling Load λ_S in the Case of Modes in the Combined Shapes of $sin\ mx$ and $sin\ 3mx$

Here, the displacement (27) becomes

$$w = \epsilon V_m^{(1)} sin\ mx + \epsilon^3\left(V_m^{(3)} sin\ mx + V_{3m}^{(3)} sin\ 3mx\right) + \epsilon^5\left(V_m^{(5)} sin\ mx + V_{3m}^{(5)} sin\ 3mx\right) + \cdots \tag{42}$$

On determining (42) at $x = \frac{\pi}{2m}$, the following is obtained

$$w = \epsilon V_m^{(1)} + \epsilon^3\left(V_m^{(3)} - V_{3m}^{(3)}\right) + \epsilon^5\left(V_m^{(5)} - V_{3m}^{(5)}\right) + \cdots \tag{43}$$

where,

$$V_m^{(3)} = \frac{-\alpha B^2}{4\theta^2}, \qquad V_{3m}^{(5)} = \frac{B^5 Q_{49}}{16\omega^2} \tag{44}$$

$$Q_{49} = (3\alpha^2 - 5\beta) \tag{45}$$

Thus, from (43) the following is obtained

$$w = \epsilon e_1 + \epsilon^3 e_2 + \epsilon^5 e_3 + \cdots \tag{46a}$$

where,

$$e_1 = B, \qquad e_2 = \frac{3\alpha B^3}{4\theta^2}Q_{51}, \qquad Q_{51} = \left[4\left(\frac{\theta}{\omega}\right)^2 - 1\right] \tag{46b}$$

$$e_3 = B^5 Q_{46} Q_{52}, \qquad Q_{52} = 1 - \frac{Q_{49}}{16 Q_{46}\omega^2} \tag{46c}$$

The series (46a) is now reversed in the form

$$\epsilon = f_1 w + f_2 w^3 + f_3 w^5 + \cdots \tag{46d}$$

where,

$$f_1 = \frac{1}{e_1}, \qquad f_2 = \frac{-e_2}{e_1^4}, \qquad f_3 = \frac{3e_2^2 - e_1 e_3}{e_1^7}$$

After simplification, we have

$$f_1 = \frac{1}{B}, \qquad f_2 = \frac{3\alpha Q_{51}}{4\theta^2 B}, \qquad f_3 = \frac{Q_{46}Q_{52}Q_{53}}{B}$$

where,

$$Q_{53} = \left(\frac{27\alpha^2 Q_{51}^2}{16\theta^4 Q_{46}Q_{52}} - 1\right) \tag{46e}$$

The maximization (28) is next invoked to yield the static buckling load λ_S (through (46d)) and through the equation

$$f_1 + 3f_2 w_{ac}^2 + 5f_3 w_{ac}^4 = 0 \tag{47a}$$

Where w_{ac} is the value of w for the displacement to have a maximum in the case of modes in the combined shapes of sinmx and sin3mx.

Thus, the following is obtained

$$w_{ac}^2 = \frac{-3f_2 \pm \sqrt{9f_2^2 - 20f_1 f_3}}{10f_3}$$
$$= \frac{3\alpha Q_{51}}{40\theta^2 Q_{46}Q_{52}Q_{53}}\left[-1 - \left\{1 - \frac{320 Q_{46}Q_{52}Q_{53}B\theta^4}{81\alpha^2 Q_{51}^2}\right\}^{\frac{1}{2}}\right] \tag{47b}$$

where, the negative square root sign in (47b) has been taken. Therefore, the following is obtained

$$w_{ac} = \left[\frac{3\alpha Q_{51}}{40\theta^2 Q_{46}Q_{52}Q_{53}}\left[-1 - \left\{1 - \frac{320 Q_{46}Q_{52}Q_{53}B\theta^4}{81\alpha^2 Q_{51}^2}\right\}^{\frac{1}{2}}\right]\right]^{\frac{1}{2}} \tag{47c}$$

To get the static buckling load in this case, (46d) is determined at static buckling to get

$$\epsilon = w_{ac}[f_1 + w_{ac}^2(f_2 + w_{ac}^2 f_3)] \tag{48a}$$

On multiplying (48a) by 5 and substituting for $5f_3 w_{ac}^4$ from (47a) and simplifying, the following is obtained

$$5m^2 \underline{a}_m \epsilon \lambda_S = 4w_{ac}(m^4 - 2m^2\lambda_S + 1)\left[1 + \frac{3\alpha w_{ac}^2 Q_{51}}{8\theta^2}\right] \tag{48b}$$

The result (48b) is implicit in the load parameter λ_S throughQ_{46}, Q_{51}, Q_{53} and θ.

6 Solution of the Dynamic Problem (5) – (8)

The full equations are here recast as

$$w_{,tt} + 2\epsilon^2 w_{,t} + w_{,xxxx} + 2\lambda f(t) w_{,xx} + w + \alpha w^3 - \beta w^5 = -2\lambda\epsilon f(t)\frac{d^2\varpi}{dx^2}, \quad t > 0, \tag{49a}$$

$$0 < x < \pi \tag{49b}$$

$$w(x,0) = w_{,t}(x,0) = 0, \quad 0 < x < \pi \tag{49c}$$

$$w = w_{,xx} = 0 \; at \; x = 0, \pi, \;\; t \geq 0 \tag{49d}$$

Let

$$\tau = \epsilon^2 t, \qquad w(x,t) = U(x,t,\tau,\epsilon) \tag{50a}$$

$$\therefore \; w_{,t} = U_{,t} + \epsilon^2 U_{,\tau}; \qquad w_{,tt} = U_{,tt} + 2\epsilon^2 U_{,t\tau} + \epsilon^4 U_{,\tau\tau} \tag{50b}$$

Substituting these in the governing equation of motion, yields

$$\left(U_{,tt} + 2\epsilon^2 U_{,t\tau} + \epsilon^4 U_{,\tau\tau}\right) + 2\epsilon^2\left(U_{,t} + \epsilon^2 U_{,\tau}\right) + U_{,xxxx} + 2\lambda U_{,xx} + U + \alpha U^3 - \beta U^5 = -2\lambda\epsilon \frac{d^2\overline{\omega}}{dx^2} \tag{51}$$

Let

$$U(x,t,\tau,\epsilon) = \sum_{i=1}^{\infty} U^{(i)} \epsilon^i \tag{52}$$

The following equations are obtained

$$O(\epsilon): LU^{(1)} \equiv U^{(1)}_{,tt} + U^{(1)}_{,xxxx} + 2\lambda U^{(1)}_{,xx} + U^{(1)} = -2\lambda\epsilon \frac{d^2\varpi}{dx^2} \tag{53}$$

$$O(\epsilon^2): \; LU^{(2)} = 0 \tag{54}$$

$$O(\epsilon^3): \; LU^{(3)} = -2\left(U^{(1)}_{,t\tau} + U^{(1)}_{,t}\right) - \alpha\left(U^{(1)}\right)^3 \tag{55}$$

$$O(\epsilon^4): \; LU^{(4)} = -3\alpha\left(U^{(1)}\right)^2 U^{(2)} - 2\left(U^{(2)}_{,t\tau} + U^{(2)}_{,t}\right) \tag{56}$$

$$O(\epsilon^5): \; LU^{(5)} = -3\alpha\left[\left(U^{(1)}\right)^2 U^{(3)} + U^{(1)}\left(U^{(2)}\right)^2\right] + \beta\left(U^{(1)}\right)^5 - 2\left(U^{(3)}_{,t\tau} + U^{(3)}_{,t}\right) \tag{57}$$

etc.

The initial conditions evaluated at $t = \tau = 0$ are

$$U^{(1)}(x,0,0) = 0, i = 1,2,3,\dots \tag{58a}$$

$$O(\epsilon): U^{(1)}_{,t} = 0 \tag{58b}$$

$$O(\epsilon^2): \; U^{(2)}_{,t} = 0 \tag{58c}$$

$$O(\epsilon^3): \; U^{(3)}_{,t} + U^{(1)}_{,\tau} = 0 \tag{58d}$$

$$O(\epsilon^4): \; U^{(4)}_{,t} + U^{(2)}_{,\tau} = 0 \tag{58e}$$

$$O(\epsilon^5): \; U^{(5)}_{,t} + U^{(3)}_{,\tau} = 0 \tag{58f}$$

$$U^{(i)} = U^{(i)}_{,xx} = 0 \; at \; x = 0, \pi \,; \quad i = 1,2,3,\dots \tag{58g}$$

Let

$$\underline{\omega} = \underline{a}_m sin\, mx, m = 1, 2, 3, \dots \text{ and let}$$

$$U^{(i)}(x, t, \tau) = \sum_{n=1}^{\infty} U_n^{(i)}(t, \tau) sin\, mx \tag{59}$$

On substituting into (53), the result is

$$\sum_{n=1}^{\infty} U_{n,tt}^{(1)} + (n^4 - 2n^2\lambda + 1)\, U_n^{(1)} = 2\underline{a}_m \lambda m^2 sinmx$$

For $n = m$, the result gives

$$U_{m,tt}^{(1)} + \theta^2 U_m^{(1)} = 2\underline{a}_m \lambda m^2 \tag{60a}$$

$$U_m^{(1)}(0,0) = U_{m,t}^{(1)}(0,0) = 0 \tag{60b}$$

The solution of (60a, b) is

$$U_m^{(1)}(t, \tau) = \alpha_1(\tau) cos\, \theta t + \gamma_1(\tau) sin\, \theta t + B \tag{60c}$$

$$B = \left(\frac{2\underline{a}_m \lambda m^2}{m^4 - 2m^2\lambda + 1}\right) = \frac{2\underline{a}_m \lambda m^2}{\theta^2} \tag{60d}$$

where, as in (21a),

$$\theta^2 = (m^4 - 2m^2\lambda + 1) > 0 \;\; \forall\, m. \tag{60e}$$

$$\therefore \;\; \alpha_1(0) = -B, \qquad \gamma_1(0) = 0 \tag{60f}$$

Substituting (59) into (54) (for $i = 2$) and for $n = m$, the following is obtained (using (58a, c))

$$U_m^{(2)}(t, \tau) = \alpha_2(\tau) cos\, \theta t + \gamma_2(\tau) sin\, \theta t \tag{61a}$$

$$\therefore \;\; \alpha_2(0) = \gamma_2(0) = 0 \tag{61b}$$

The next substitution into (55) requires the simplification

$$U_m^{(1)^3} = (\alpha_1(\tau) cos\, \theta t + \gamma_1(\tau) sin\, \theta t + B)^3 = \left(3B\alpha_1^2 + B^3 + \frac{3B\gamma_1^2}{2}\right) + \left\{\frac{3\alpha_1^3}{4} + 3\alpha_1\left(B^2 + \frac{\gamma_1^2}{2}\right) - 3\alpha 1\gamma 124 \cos\theta t + 3\alpha 12\gamma 14 + 3B2\gamma 1 + \gamma 134 \sin\theta t + 3B\alpha 12 - B\gamma 122 \cos 2\theta t + 3\alpha 1\gamma 1 B sin\, 2\theta t + \left(\frac{\alpha_1^3}{4} - \frac{3\alpha_1\gamma_1^2}{4}\right) \cos 3\theta t + \left(\frac{3\alpha_1^2\gamma_1}{2} - \frac{\gamma_1^3}{4}\right) sin\, 3\theta t \tag{62}$$

Substituting (59) into (55) and for $n = m$, the following is obtained

$$U_{m,tt}^{(3)} + \theta^2 U_m^{(3)} = -2\left(U_{m,t\tau}^{(3)} + \theta^2 U_{m,t}^{(3)}\right) - 3\alpha U_m^{(1)^3} \tag{63a}$$

$$U_m^{(3)}(0,0) = U_{m,t}^{(3)}(0,0) + U_{m,\tau}^{(1)}(0,0) = 0 \tag{63b}$$

However, for $n = 3m$ in the substitution into (55), the resultant equation is

$$U_{3m,tt}^{(3)} + \omega^2 U_{3m}^{(3)} = \alpha {U_m^{(1)}}^3 \qquad (64a)$$

$$U_{3m}^{(3)}(0,0) = \quad U_{3m,t}^{(3)}(0,0) = 0 \qquad (64b)$$

$$\omega^2 = (81m^4 - 18m^2\lambda + 1) > 0 \ \forall \ m. \qquad (64c)$$

Now, substituting from (60c, d) and (62) into (63a) and ensuring a uniformly valid solution in t by equating separately, on the right hand side, the coefficients of $cos\ \theta t$ and $sin\ \theta t$ to zero, it is obtained, for coefficient of $cos\ \theta t$:

$$-2\theta(\gamma_1' + \gamma_1) - 3\alpha\left\{\frac{3\alpha_1^3}{4} + 3\alpha_1\left(B^2 + \frac{\gamma_1^2}{2}\right)\right\} + \frac{9\alpha_1\gamma_1^2}{4} = 0 \qquad (65a)$$

and for coefficient of $sin\ \theta t$, the following is obtained

$$2\theta(\alpha_1' + \alpha_1) - 3\alpha\left\{\frac{3\alpha_1^2\gamma_1}{4} + 3\left(B^2\gamma_1 + \frac{\gamma_1^3}{4}\right)\right\} = 0 \qquad (65b)$$

Simplifying (65a, b) further, the following is obtained

$$2\theta(\gamma_1' + \gamma_1) + 9\alpha\alpha_1 K = 0 \qquad (65c)$$

$$2\theta(\alpha_1' + \alpha_1) - 9\alpha\gamma_1 K = 0 \qquad (65d)$$

where,

$$K = \frac{\alpha_1^2}{4} + B^2 + \frac{\gamma_1^2}{4} \qquad (65e)$$

Multiplying (65c) by γ_1 and (65d) by α_1 and adding, followed by simplification, yields

$$\frac{\frac{1}{2}\frac{d}{d\tau}(\gamma_1^2 + \alpha_1^2)}{\gamma_1^2 + \alpha_1^2} + 1 = 0 \qquad (65f)$$

The solution of (65f), using (60f), yields

$$(\gamma_1^2 + \alpha_1^2) = B^2 e^{-2\tau} \qquad (65g)$$

It should be noted that α_1 and γ_1 are not going to be solved explicitly because every information needed later about $\alpha_1(\tau)$ and $\gamma_1(\tau)$ can be easily obtained from (65c – g). For example, the following results are needed

$$\alpha_1'(0) = -\alpha_1(0) = B, \qquad \alpha_1''(0) = \frac{2025\alpha^2 B^5}{65\theta^2} - B \qquad (65h)$$

$$\gamma_1'(0) = \frac{45\alpha B^3}{8\theta}, \qquad \gamma_1''(0) = \frac{-27\alpha B^3}{2\theta} \qquad (65i)$$

The remaining equation in the substitution into (63a) is

$$U_{m,tt}^{(3)} + \theta^2 U_m^{(3)} = -3\alpha[r_0 + r_1 cos\ 2\theta t + r_2 sin\ 2\theta t + r_3 cos\ 3\theta t + r_4 sin\ 3\theta t\ ] \qquad (66a)$$

$$U_m^{(3)}(0,0) = 0, \qquad U_{m,t}^{(3)}(0,0) + U_{m,\tau}^{(1)}(0,0) = 0$$

where,

$$r_0 = \left(3B\alpha_1^2 + B^3 + \frac{3B\gamma_1^2}{2}\right), \qquad r_0(0) = 4B^3 \qquad (66b)$$

$$r_1 = 3\left(B\alpha_1^2 - \frac{B\gamma_1^2}{2}\right), \qquad r_1(0) = 3B^3 \qquad (66c)$$

$$r_2 = 3\alpha_1\gamma_1 B, \qquad r_2(0) = 0; \qquad r_3 = \frac{\alpha_1^3}{4} - \frac{3\alpha_1\gamma_1^2}{4}, \qquad r_3(0) = \frac{-B^3}{4} \qquad (66d)$$

$$r_4 = \gamma_1\left(\frac{3\alpha_1^2}{2} - \frac{\gamma_1^2}{4}\right), \qquad r_4(0) = 0 \qquad (66e)$$

The solution of (66a – e) is

$$U_m^{(3)} = \alpha_{12}(\tau)cos\,\theta t + \beta_{12}(\tau)sin\,\theta t - 3\alpha\left[\frac{r_0}{\theta^2} - \frac{r_1 os\,2\theta t}{3\theta^2} - \frac{r_2 sin\,2\theta t}{3\theta^2} - \frac{1}{8\theta^2}(\,r_3 cos\,3\theta t + r4sin\,3\theta t \right. \qquad (67a)$$

where,

$$\alpha_{12}(0) = 3\alpha\left[\frac{r_0}{\theta^2} - \frac{r_1}{3\theta^2} - \frac{r_3}{8\theta^2}\right]\Big|_{\tau=0} = \frac{291\alpha B^3}{32\theta^2}; \qquad \beta_{12}(0) = \frac{-B}{\theta} \qquad (67b)$$

The following terms will be useful later:

$$r_1'(0) = -6B^3, \qquad r_2'(0) = \frac{-135\alpha B^5}{8\theta}, \qquad r_3'(0) = \frac{3B^3}{4}, \qquad r_4'(0) = \frac{135\alpha B^5}{16\theta}$$

Now, going back to the substitution in (55) to the case $n = 3m$, the resultant equation is

$$U_{3m,tt}^{(3)} + \omega^2 U_{3m}^{(3)} = \alpha[r_0 + r_5 cos\theta t + r_6 sin\theta t + r_1 cos\,2\theta t + r_2 sin\,2\theta t + r_3 cos\,3\theta t + r4sin\,3\theta t \qquad (68a)$$

$$U_{3m}^{(3)}(0,0) = U_{3m,t}^{(3)}(0,0) = 0 \qquad (68b)$$

where,

$$r_5 = \left[\frac{\alpha_1^3}{4} + 3\alpha_1\left(B^2 + \frac{\gamma_1^2}{4}\right) - \frac{3\alpha_1\gamma_1^2}{4}\right], \qquad r_5(0) = \frac{-B^3}{4}$$

$$r_6 = \left[\frac{3\alpha_1^2\gamma_1}{4} + 3\left(B^2\gamma_1 + \frac{\gamma_1^3}{4}\right)\right], \qquad r_6(0) = 0$$

$$\omega^2 = (81m^4 - 18m^2\lambda + 1) > 0 \;\; \forall\, m. \qquad (68c)$$

It also follows that

$$r_5'(0) = \frac{21B^3}{4}, \qquad r_6'(0) = \frac{6755\alpha B^5}{32\theta}$$

The solution of (68a, b) is

$$U_{3m}^{(3)} = \alpha_3 cos\ \omega t + \beta_3 sin\ \omega t + \alpha\left[\frac{r_0}{\omega^2} + \left(\frac{r_5 cos\theta t + r_6 sin\theta t}{\omega^2 - \theta^2}\right) + \left(\frac{r_1 cos\ 2\theta t + r_2 sin\ 2\theta t}{\omega^2 - 4\theta^2}\right) + \left(\frac{r_3 cos\ 3\theta t + r_4 sin\ 3\theta t}{\omega^2 - 9\theta^2}\right)\right] \qquad (69a)$$

$$\alpha_3(0) = -\alpha B^3 \Omega_1, \quad \Omega_1 = \left[\frac{4}{\omega^2} - \frac{15}{4(\omega^2 - \theta^2)} + \frac{3}{\omega^2 - 4\theta^2} + \frac{1}{4(\omega^2 - 9\theta^2)}\right] \qquad (69b)$$

$$\beta_3(0) = 0 \qquad (69c)$$

The next substitution into (56) requires the following simplification

$$U_m^{(3)^2} U_m^{(2)} = [r_{18} + r_{19} cos\theta t + r_{20} sin\theta t + r_{21} cos\ 2\theta t + r_{22} sin\ 2\theta t + r_{23} cos\ 3\theta t + r_{24} sin\ 3\theta t\ ] \qquad (70a)$$

where,

$$r_{18} = B(\alpha_1\alpha_2 + \gamma_1\gamma_2), \qquad r_{18}(0) = 0 \qquad (70b)$$

$$r_{19} = \left[\left(\frac{\alpha_1^2}{2} + \frac{\gamma_1^2}{2} + B\right)\alpha_2 + \frac{\alpha_1\gamma_1\gamma_2}{2} + \frac{\gamma_2}{4}(\alpha_1^2 - \gamma_1^2)\right], \qquad r_{19}(0) = 0 \qquad (70c)$$

$$r_{20} = \left[\left(\frac{\alpha_1^2}{2} + \frac{\gamma_1^2}{2} + B^2\right)\gamma_2 + \frac{\alpha_1\alpha_2\gamma_1}{2} + \frac{\gamma_2}{4}(\alpha_1^2 - \gamma_1^2)\right], \qquad r_{20}(0) = 0 \qquad (70d)$$

$$r_{21} = B(\alpha_1\alpha_2 - \gamma_1\gamma_2), \qquad r_{21}(0) = 0 \qquad (70e)$$

$$r_{22} = B(\alpha_1\gamma_2 + \alpha_2\gamma_1), \qquad r_{22}(0) = 0 \qquad (70f)$$

$$r_{23} = \left[\frac{(\alpha_1^2 - \gamma_1^2)\alpha_2}{4} - \frac{\alpha_1\gamma_1\gamma_2}{4}\right], \qquad r_{23}(0) = 0 \qquad (70g)$$

$$r_{24} = \left[\frac{\alpha_1\alpha_2\gamma_1}{2} + \frac{\gamma_2}{4}(\alpha_1^2 - \gamma_1^2)\right], \qquad r_{24}(0) = 0 \qquad (70h)$$

Now substituting into (56), yields

$$U_{,tt}^{(4)} + U_{,xxxx}^{(4)} + 2\lambda U_{,xx}^{(4)} + U^{(4)} = -2\left(U_{m,t\tau}^{(2)} + U_{m,t}^{(2)}\right) sin\ mx - \frac{3\alpha}{4} U_m^{(1)^2} U_m^{(2)} (3 sin\ mx - sin\ 3mx\ ) \qquad (71)$$

After substituting (59) into (71), for $i = 4$, and for $n = m$, the following are obtained

$$U_{m,tt}^{(4)} + \theta^2 U_m^{(4)} = -2\left(U_{m,t\tau}^{(2)} + U_{m,t}^{(2)}\right) - 9\alpha U_m^{(1)^2} U_m^{(2)} \qquad (72a)$$

$$U_m^{(4)}(0,0) = 0, \qquad U_{m,t}^{(4)}(0,0) + U_{m,\tau}^{(2)}(0,0) = 0 \qquad (72b)$$

However, for $n = 3m$, the following are obtained

$$U_{3m,tt}^{(4)} + \omega^2 U_{3m}^{(4)} = \frac{3\alpha}{4} U_m^{(1)^2} U_m^{(2)} \quad (72c)$$

$$U_{3m}^{(4)}(0,0) = \quad U_{3m,t}^{(4)}(0,0) = 0 \quad (72d)$$

Now, on substituting into (72a) for $U_m^{(2)}$ from (61a) as well as for $U_m^{(1)^2} U_m^{(2)}$ from (70a) and ensuring a uniformly valid solution in t by separately equating to zero the coefficients of $cos\,\theta t$ and $sin\,\theta t$, the following are obtained, respectively

$$\gamma_2' + \gamma_2 = \frac{-9\alpha r_{19}}{8\theta} \quad and \quad \alpha_2' + \alpha_2 = \frac{-9\alpha r_{20}}{4} \quad (73a)$$

Solving (73a) gives

$$\gamma_2 = -e^{-\tau}\int_0^{\tau} \frac{9\alpha r_{19}}{8\theta} d\tau, \quad \alpha_2 = \quad e^{-\tau}\int_0^{\tau} \frac{9\alpha r_{20}}{8\theta} d\tau \quad (73b)$$

Observation shows that

$$\gamma_2'(0) = \alpha_2' = \gamma_2''(0) = \alpha_2''(0) = 0$$

Infact, all derivatives of γ_2 and α_2 evaluated at $\tau = 0$ vanish. Thus, without loss of generality,

$$\gamma_2(\tau) = \alpha_2(\tau) \equiv 0$$

Hence, the following are obtained

$$U^{(2)} = U_m^{(2)}(t,\tau) = 0 \quad (73c)$$

The remaining equation in (72a) is

$$U_{m,tt}^{(4)} + \theta^2 U_m^{(4)} = -\frac{9\alpha}{4}[r_{21} cos\,2\theta t + r_{22} sin\,2\theta t + r_{23}\, cos\,cos\,3\theta t + r_{24} sin\,3\theta t\,] \quad (74a)$$

which is now solved together with (72b) to yield

$$U_m^{(4)} = q_1(\tau) cos\,\theta t + q_2(\tau) sin\,\theta t - \frac{9\alpha}{4}\left[-\frac{1}{3\theta^2}(r_{21} cos\,2\theta t + r_{22} sin\,2\theta t\,) - \frac{1}{8\theta^2}(r_{23} cos\,3\theta t + r_{24} sin\,3\theta t\,)\right] \quad (74b)$$

$$q_1(0) = 0 = q_2(0) \quad (74c)$$

On substituting from (70a) into (72c) and solving, the following are obtained

$$U_{3m}^{(4)} = \alpha_4(\tau) cos\,\omega t + \gamma_4(\tau) sin\,\omega t + \frac{3\alpha}{4}\left[\frac{r_{18}}{\omega^2} + \left(\frac{r_{19} cos\theta t + r_{20} sin\theta t}{\omega^2 - \theta^2}\right) + \left(\frac{r_{21} cos\,2\theta t + r_{22} sin\,2\theta t}{\omega^2 - 4\theta^2}\right) + \left(\frac{r_{23} cos\,3\theta t + r_{24} sin\,3\theta t}{\omega^2 - 9\theta^2}\right)\right] \quad (75a)$$

$$\alpha_4(0) = \gamma_4(0) = 0 \quad (75b)$$

Thus,

$$U^{(4)} = U_m^{(4)} \sin mx + U_{3m}^{(4)} \sin 3mx \tag{75c}$$

The next substitution into (57), using (73c), needs the following simplications:

$$\begin{aligned} {U_m^{(1)}}^5 &= (\alpha_1 cos\ \theta t\ + \gamma_1 sin\ \theta t\ + B)^5 \\ &= r_7 + r_8 cos\theta t + r_9 sin\theta t + r_{10} cos\ 2\theta t + r_{11} sin\ 2\theta t + r_{12} cos\ 3\theta t + r_{13} sin\ 3\theta t \\ &+ r_{14} cos\ 4\theta t + r_{15} sin\ 4\theta t\ + r_{16} cos\ 5\theta t + r_{17} sin\ 5\theta t \end{aligned} \tag{75d}$$

where,

$$\begin{aligned} r_7 &= \frac{15\alpha_1^4 B}{8} + 5\alpha_1^2\left(B^3 + \frac{3\gamma_1^2 B}{4}\right) - \frac{15\alpha_1\gamma_1^2}{16} + \left(\frac{\gamma_1^2}{2} + B^2\right)\left(B^3 + \frac{\gamma_1^2 B}{2}\right) + 3B\gamma_1\left(\frac{\gamma_1^3}{4} + B^2\gamma_1\right) \\ &+ \frac{3\gamma_1^4 B}{8} \end{aligned} \tag{75e}$$

$$\begin{aligned} r_8 = \frac{5\alpha_1^5}{8} &+ \frac{10\alpha_1^3 B}{4}\left(\frac{\gamma_1^2}{2} + 3B^2\right) \\ &+ 5\alpha_1\left\{\left(\frac{3\gamma_1^4}{8} + 3B^2\gamma_1 + B^4\right) - \frac{1}{2}\left(\frac{3\gamma_1^4}{8} + 3\gamma_1^2 B^2\right) - \frac{B\gamma_1^4}{4}\right\} \end{aligned} \tag{75f}$$

$$\begin{aligned} r_9 = \frac{-5\alpha_1^4\gamma_1}{16} &- \frac{10\alpha_1^3\gamma_1 B}{4} + 5\alpha_1^2\left(\frac{\gamma_1^3}{2} - \frac{5B^2\gamma_1}{8}\right) + 3\left(\frac{\gamma_1^2}{2} + B^2\right)\left(\frac{\gamma_1^3}{4} + B^2\gamma_1\right) \\ &+ 2\gamma_1 B\left(B^3 + \frac{\gamma_1^2 B}{2}\right) + \frac{B^2\gamma_1^3}{2} + \frac{3}{4}\left(\frac{\gamma_1^3}{4} + B^2\gamma_1\right) + \frac{\gamma_1^5}{16} \end{aligned} \tag{75g}$$

$$r_{10} = \frac{5\alpha_1^4 B}{4} + 5\alpha_1^2 B^3 - \frac{3}{2}\left(\frac{\gamma_1^2}{2} + B^2\right)\gamma_1^2 B - 3B\gamma_1\left(\frac{\gamma_1^3}{4} + B^2\gamma_1\right) - \left(B^3 + \frac{3\gamma_1^2 B}{2}\right)\frac{\gamma_1^2}{2} \tag{75h}$$

$$r_{11} = \frac{15\alpha_1^3\gamma_1 B}{4} + 5\alpha_1\left\{\left(\frac{11B^2\gamma_1}{4} + \frac{13B\gamma_1^3}{8}\right)\right\} \tag{75i}$$

$$r_{12} = \frac{5\alpha_1^5}{16} + \frac{5\alpha_1^3}{2}\left(3B^2 - \frac{\gamma_1^3}{4}\right) + 5\alpha_1\left\{-\frac{1}{2}\left(\frac{3\gamma_1^4}{8} + 3\gamma_1^2 B^2\right) + \frac{B\gamma_1^4}{4}\right\} \tag{75j}$$

$$\begin{aligned} r_{13} = \frac{15\alpha_1^4\gamma_1}{8} &+ \frac{5\alpha_1^3\gamma_1 B}{2} + 5\alpha_1^2\left(\frac{3\gamma_1^3}{8} + \frac{5}{4}B^2\gamma_1\right) - \frac{1}{4}\left(\frac{\gamma_1^2}{2} + B^2\right)\gamma_1^3 - \frac{B^3\gamma_1^2}{2} \\ &- \frac{3}{4}\left(\frac{\gamma_1^3}{4} + B^2\gamma_1\right)\gamma_1^2 \end{aligned} \tag{75k}$$

$$r_{14} = \frac{5\alpha_1^4 B}{4} - \frac{15\alpha_1^2\gamma_1^2 B}{4} + \frac{3\gamma_1^4 B}{8} \tag{75l}$$

$$r_{15} = \frac{-5\alpha_1\gamma_1^3 B}{2} \tag{75m}$$

$$r_{16} = \frac{\alpha_1^5}{8} - \frac{5\alpha_1^3\gamma_1^2}{8} + \frac{15\alpha_1\gamma_1^4}{16} \tag{75n}$$

$$r_{17} = \frac{5\alpha_1^4\gamma_1}{8} - \frac{15\alpha_1^2\gamma_1^2 B}{8} + \frac{\gamma_1^5}{16} \tag{75o}$$

where,

$$r_7(0)=\frac{63B^5}{8},\quad r_8(0)=\frac{-105B^5}{8},\quad r_9(0)=0,\quad r_{10}(0)=\frac{25B^5}{4},\quad r_{11}(0)=0,\quad r_{12}(0)=\frac{-125B^5}{16},\quad r_{13}(0)=0,\quad r_{14}(0)=\frac{5B^5}{4},\quad r_{15}(0)=0,\quad r_{16}(0)=\frac{-B^5}{8},\quad r_{17}(0)=0$$

The following simplifications are also needed,

$$U_m^{(1)^2}U_m^{(3)}=r_{25}+r_{26}cos\theta t+r_{27}sin\theta t+r_{28}cos\,2\theta t+r_{29}sin\,2\theta t+r_{30}cos\,3\theta t+r_{31}\,sin\,sin\,3\theta t+r_{32}cos\,4\theta t+r_{33}sin\,4\theta t+r_{34}cos\,5\theta t+r_{35}sin\,5\theta t \qquad (76a)$$

where,

$$r_{25}=\left\{\frac{1}{2}(\alpha_1^2+\gamma_1^2)+B^2\right\}\left(\frac{-3\alpha r_0}{\theta^2}\right)+B\alpha_1\alpha_{12}+B\gamma_1\beta_{12}+\alpha\left(\frac{\alpha_1^2-\gamma_1^2}{2\theta^2}\right)\gamma_1 \qquad (76b)$$

$$r_{26}=\left\{\frac{1}{2}(\alpha_1^2-\gamma_1^2)+B^2\right\}\alpha_{12}-\frac{6\alpha r_0B\alpha_1}{\theta^2}+\frac{B\alpha_1\alpha r_1}{\theta^2}+\frac{B\alpha_2\alpha\gamma_{12}}{\theta^2}+\frac{1}{4}(\alpha_1^2-\gamma_1^2)\alpha_{12}+\frac{3\alpha(\alpha_1^2-\gamma_1^2)r_3}{32\theta^2} \qquad (76c)$$

$$r_{27}=\left\{\frac{1}{2}(\alpha_1^2-\gamma_1^2)+B^2\right\}\beta_{12}+\frac{B\alpha_1\alpha r_2}{\theta^2}-\frac{6\alpha r_0B\gamma_1}{\theta^2}-\frac{B\alpha_1\alpha\gamma_1}{\theta^2}-\frac{3B\alpha\gamma_1r_3}{8\theta^2}-\frac{1}{4}(\alpha_1^2-\gamma_1^2)\beta_{12}+\frac{3\alpha(\alpha_1^2-\gamma_1^2)r_4}{32\theta^2} \qquad (76d)$$

In the same vein, the following simplifications are also needed

$$U_m^{(1)^2}U_{3m}^{(3)}=r_{36}+r_{37}cos\theta t+r_{38}sin\theta t+r_{39}cos\,2\theta t+r_{40}sin\,2\theta t+r_{41}cos\,3\theta t+r_{42}sin\,3\theta t+r_{43}cos\,4\theta t+r_{44}sin\,4\theta t+r_{45}cos\,5\theta t+r_{46}sin\,5\theta t+r_{47}cos(\theta+\omega)t+r_{48}sin(\theta+\omega)t+r_{49}cos\,(\theta-\omega)t+r_{50}sin\,(\theta-\omega)t+r_{51}\,cos\,cos\,(2\theta+\omega)t+r_{52}\,sin\,sin\,(2\theta+\omega)t+r_{53}\,cos\,cos\,(\omega-2\theta)t+r_{54}sin\,(\omega-2\theta)t+r_{55}cos\,\omega t+r_{56}sin\,\omega t \qquad (77a)$$

where,

$$r_{36}=\left(\frac{\alpha r_0}{\omega^2}\right)\left\{\frac{1}{2}(\alpha_1^2-\gamma_1^2)+B^2\right\}+\frac{2B\alpha\alpha_1r_0}{\omega^2}+\frac{B\alpha\alpha_1r_5}{\omega^2-\theta^2}+\frac{B\alpha\gamma_1r_6}{\omega^2-\theta^2}+\frac{\alpha(\alpha_1^2-\gamma_1^2)r_1}{4(\omega^2-\theta^2)} \qquad (77b)$$

$$r_{37}=\left(\frac{\alpha r_5}{\omega^2-\theta^2}\right)\left\{\frac{1}{2}(\alpha_1^2-\gamma_1^2)+B^2\right\}+\frac{2B\alpha\alpha_1r_1}{\omega^2-4\theta^2}+\frac{B\alpha\gamma_1r_2}{\omega^2-4\theta^2}+\frac{1}{4}\left(\frac{\alpha r_5}{\omega^2-\theta^2}\right)(\alpha_1^2-\gamma_1^2)-\frac{\alpha(\alpha_1^2-\gamma_1^2)r_3}{4(\omega^2-9\theta^2)} \qquad (77c)$$

$$r_{38}=\left(\frac{\alpha r_6}{\omega^2-\theta^2}\right)\left\{\frac{1}{2}(\alpha_1^2-\gamma_1^2)+B^2\right\}+\frac{B\alpha\gamma_1r_0}{\omega^2}-\frac{B\alpha\gamma_1r_1}{\omega^2-4\theta^2}-\frac{\alpha(\alpha_1^2-\gamma_1^2)r_6}{4(\omega^2-\theta^2)}+\frac{\alpha(\alpha_1^2-\gamma_1^2)r_4}{4(\omega^2-9\theta^2)} \qquad (77d)$$

$$r_{39} = \left(\frac{\alpha r_1}{\omega^2 - 4\theta^2}\right)\left\{\frac{1}{2}(\alpha_1^2 - \gamma_1^2) + B^2\right\} + \frac{B\alpha\alpha_1 r_5}{\omega^2 - \theta^2} + \frac{B\alpha\alpha_1 r_3}{\omega^2 - 9\theta^2} - \frac{B\alpha\gamma_1 r_6}{\omega^2 - \theta^2} + \frac{B\alpha\gamma_1 r_4}{\omega^2 - 9\theta^2} + \frac{1}{4}(\alpha_1^2 - \gamma_1^2)\left(\frac{\alpha r_0}{\omega^2}\right) \quad (77e)$$

$$r_{40} = \left(\frac{\alpha r_2}{\omega^2 - 4\theta^2}\right)\left\{\frac{1}{2}(\alpha_1^2 - \gamma_1^2) + B^2\right\} + \frac{B\alpha\alpha_1 r_6}{\omega^2 - \theta^2} + \frac{B\alpha\alpha_1 r_2}{\omega^2 - 4\theta^2} + \frac{B\alpha\alpha_1 r_4}{\omega^2 - 9\theta^2} + \frac{B\alpha\alpha_5\gamma_1}{\omega^2 - \theta^2} - \frac{B\alpha\gamma_1 r_3}{\omega^2 - 9\theta^2} \quad (77f)$$

$$r_{41} = \left(\frac{\alpha r_3}{\omega^2 - 9\theta^2}\right)\left\{\frac{1}{2}(\alpha_1^2 - \gamma_1^2) + B^2\right\} + \frac{B\alpha\gamma_1\alpha_1}{\omega^2 - 4\theta^2} - \frac{B\alpha\gamma_1 r_4}{\omega^2 - 9\theta^2} + \frac{\alpha(\alpha_1^2 - \gamma_1^2)r_5}{4(\omega^2 - 4\theta^2)} - \frac{B\alpha\gamma_1 r_2}{\omega^2 - 4\theta^2} \quad (77g)$$

$$r_{42} = \left(\frac{\alpha r_4}{\omega^2 - 9\theta^2}\right)\left\{\frac{1}{2}(\alpha_1^2 - \gamma_1^2) + B^2\right\} + \frac{B\alpha\gamma_1 r_1}{\omega^2 - 4\theta^2} + \frac{1}{4}(\alpha_1^2 - \gamma_1^2)\left(\frac{\alpha r_6}{\omega^2 - \theta^2}\right) \quad (77h)$$

$$r_{43} = \frac{B\alpha\alpha_1 r_3}{\omega^2 - 9\theta^2} + \frac{\alpha(\alpha_1^2 - \gamma_1^2)r_1}{4(\omega^2 - 9\theta^2)} \quad (77i)$$

$$r_{44} = \frac{B\alpha\alpha_1 r_4}{\omega^2 - 9\theta^2} + \frac{B\alpha\gamma_1 r_3}{\omega^2 - 9\theta^2} + \frac{\alpha(\alpha_1^2 - \gamma_1^2)r_2}{4(\omega^2 - 9\theta^2)} \quad (77j)$$

$$r_{45} = \frac{\alpha(\alpha_1^2 - \gamma_1^2)r_3}{4(\omega^2 - 9\theta^2)}, \qquad r_{46} = \frac{\alpha(\alpha_1^2 - \gamma_1^2)r_4}{4(\omega^2 - 9\theta^2)} \quad (77k)$$

$$r_{47} = -B\gamma_1\beta_3, \qquad r_{48} = B\alpha_1\beta_3 + B\gamma_1\gamma_3 \quad (77l)$$

$$r_{49} = (B\alpha_1\alpha_3 + B\gamma_1\beta_3), \qquad r_{50} = B\alpha_1\beta_3 + B\gamma_1\gamma_3 \quad (77m)$$

$$r_{51} = \frac{\alpha_3(\alpha_1^2 - \gamma_1^2)}{4}, \qquad r_{52} = \frac{\beta_3(\alpha_1^2 - \gamma_1^2)}{4} \quad (77n)$$

$$r_{53} = \frac{\alpha_3(\alpha_1^2 - \gamma_1^2)}{4}, \qquad r_{54} = \frac{\beta_3(\alpha_1^2 - \gamma_1^2)}{4} \quad (77o)$$

$$r_{55} = \alpha_3\left\{\frac{1}{2}(\alpha_1^2 + \gamma_1^2) + B^2\right\}, \qquad r_{56} = \beta_3\left\{\frac{1}{2}(\alpha_1^2 + \gamma_1^2) + B^2\right\} \quad (77p)$$

where,

$$r_{36}(0) = \alpha B^5\Omega_2, \quad \Omega_2 = \left(\frac{9}{2(\omega^2 - \theta^2)} - \frac{2}{\omega^2}\right)$$

$$r_{37}(0) = \alpha B^5\Omega_3, \quad \Omega_3 = \left(\frac{9}{8(\omega^2 - \theta^2)} + \frac{1}{16(\omega^2 - \theta^2)} - \frac{3}{\omega^2 - 4\theta^2}\right)$$

$$r_{38}(0) = 0, \qquad r_{39}(0) = \alpha B^5\Omega_4, \quad \Omega_4 = \left(\frac{9}{2(\omega^2 - \theta^2)} + \frac{15}{4(\omega^2 - \theta^2)} + \frac{1}{4(\omega^2 - 9\theta^2)} + \frac{1}{\omega^2}\right)$$

$$r_{40}(0) = 0, \qquad r_{41}(0) = \alpha B^5\Omega_4^{(1)}, \quad \Omega_4^{(1)} = \left(\frac{3}{8(\omega^2 - 9\theta^2)} - \frac{111}{32(\omega^2 - 4\theta^2)}\right)$$

$$r_{42}(0) = 0, \qquad r_{43}(0) = \alpha B^5\Omega_5, \quad \Omega_5 = \left(\frac{3}{4(\omega^2 - 4\theta^2)} + \frac{1}{4(\omega^2 - 9\theta^2)}\right)$$

$$r_{44}(0) = 0, \quad r_{45}(0) = -\alpha B^5 \Omega_6, \quad \Omega_6 = -\frac{1}{16(\omega^2 - 9\theta^2)}$$
$$r_{46}(0) = 0, \quad r_{47}(0) = 0, \quad r_{48}(0) = 0$$
$$r_{49}(0) = \alpha B^5 \Omega_1, \quad r_{50}(0) = 0, \quad r_{51}(0) = \frac{-\alpha B^5 \Omega_1}{4}$$
$$r_{52}(0) = 0, \quad r_{53}(0) = \frac{-\alpha B^5 \Omega_1}{4}, \quad r_{54}(0) = 0, \quad r_{55}(0) = \frac{-3\alpha B^5 \Omega_1}{2}, \quad r_{56}(0) = 0.$$

Now, substituting the relevant terms into (57) yields

$$\begin{aligned} U^{(5)}_{,tt} + U^{(5)}_{,xxxx} + 2\lambda U^{(5)}_{,xx} + U^{(5)} &= -2\left(U^{(3)}_{m,t\tau} + U^{(3)}_{m,t}\right) sin\, mx - \left(U^{(3)}_{3m,t\tau} + U^{(3)}_{3m,t}\right) sin 3mx \\ &- 3\alpha\left[\frac{1}{4} {U^{(1)}_m}^2 U^{(3)}_m (3 sin\, mx - sin\, 3mx) + \frac{1}{4} U^{(1)}_m U^{(3)}_m (sin\, 3mx - sin\, mx)\right] \\ &+ \frac{\beta}{16} {U^{(1)}_m}^5 [11 sin\, mx - 5 sin\, 3mx + sin\, 5mx] \end{aligned} \qquad (78a)$$

Using (59) for $n = m$, the following are obtained

$$U^{(5)}_{m,tt} + \theta^2 U^{(5)}_m = -2\left(U^{(3)}_{m,t\tau} + U^{(3)}_{m,t}\right) - \frac{3\alpha}{4}\left[3{U^{(1)}_m}^2 U^{(3)}_m - {U^{(1)}_m}^2 U^{(3)}_{3m}\right] + \frac{11\beta}{16} {U^{(1)}_m}^5 \qquad (78b)$$

$$U^{(5)}_m(0,0) = 0, \quad U^{(5)}_{m,t}(0,0) + U^{(3)}_{m,\tau}(0,0) = 0 \qquad (78c)$$

where as for $n = 3m$, the following are obtained

$$U^{(5)}_{3m,tt} + \omega^2 U^{(5)}_{3m} = -2\left(U^{(3)}_{3m,t\tau} + U^{(3)}_{3m,t}\right) - \frac{5\beta}{16} {U^{(1)}_m}^5 \qquad (78d)$$

$$U^{(5)}_{3m}(0,0) = 0, \quad U^{(5)}_{3m,t}(0,0) + U^{(3)}_{3m,\tau}(0,0) = 0 \qquad (78e)$$

For $n = 5m$, the following are obtained

$$U^{(5)}_{5m,tt} + \varphi^2 U^{(5)}_{5m} = \frac{\beta}{16} {U^{(1)}_m}^5 \qquad (78f)$$
$$U^{(5)}_{5m}(0,0) = 0, \quad U^{(5)}_{5m,t}(0,0) = 0 \qquad (78g)$$

where,

$$\varphi^2 = (625m^4 - 50m^2\lambda + 1) > 0 \;\; \forall\, m. \qquad (78h)$$

After substituting into (78b), using (75a – o) and (76a – d), and ensuring a uniformly valid solution in t by equating to zero the coefficients of $cos\,\theta t$ and $sin\,\theta t$, the following are obtained, for the coefficient of $cos\,\theta t$,

$$\beta_{12}' + \beta_{12} = \rho_1(\tau), \quad \rho_1 = \frac{1}{2\theta}\left[\alpha\left(\frac{3}{4} r_{37} - \frac{9 r_{26}}{4}\right) + \frac{11\beta r_8}{16}\right] \qquad (79a)$$

and for $sin\,\theta t$, the following are obtained

$$\alpha_{12}' + \alpha_{12} = \rho_2(\tau), \quad \rho_2 = \frac{1}{2\theta}\left[\frac{\alpha}{4}(9 r_{27} - 3 r_{38}) - \frac{11\beta r_9}{16}\right] \qquad (79b)$$

The solutions of (79a, b) are

$$\beta_{12} = e^{-\tau}\left[\int_0^\tau e^s \rho_1(s)ds\right], \quad \alpha_{12} = e^{-\tau}\left[\int_0^\tau e^s \rho_2(s)ds + \alpha_{12}(0)\right] \tag{79c}$$

Meanwhile, it follows from (79a – c) that

$$\beta_{12}'(0) = B^5\Omega_7, \qquad \alpha_{12}'(0) = -\frac{483\alpha B^3}{32\theta^2} \tag{79d}$$

where,

$$\Omega_7 = \left[\alpha^2\left\{\frac{3\Omega_3}{4} - \frac{27{,}297}{256\theta^2}\right\} - \frac{1155\beta}{128}\right] \tag{79e}$$

The remaining equation in the substitution into (78b) is

$$\begin{aligned} U_{m,tt}^{(5)} + \theta^2 U_m^{(5)} = {} & r_{57} + r_{58}cos2\theta t + r_{59}sin2\theta t + r_{60}cos\,3\theta t + r_{61}sin\,3\theta t + r_{62}cos\,4\theta t \\ & + r_{63}sin\,4\theta t + r_{64}cos\,5\theta t + r_{65}sin\,5\theta t + r_{66}cos(\theta+\omega)t + r_{67}sin(\theta+\omega)t \\ & + r_{68}cos\,(\theta-\omega)t + r_{69}sin\,(\theta-\omega)t + r_{70}cos\,(2\theta+\omega)t + r_{71}sin\,(2\theta+\omega)t \\ & + r_{72}cos\,(\omega-2\theta)t + r_{73}sin\,(\omega-2\theta)t + r_{74}cos\,\omega t + r_{75}sin\,\omega t \end{aligned} \tag{80a}$$

where,

$$r_{57} = \left[\frac{-9\alpha r_{25}}{4} + \frac{3\alpha r_{36}}{4} + \frac{11\beta r_7}{16}\right] \tag{80b}$$

$$r_{58} = \left[\frac{-9\alpha r_{28}}{4} + \frac{3\alpha r_{39}}{4} + \frac{11\beta r_{10}}{16}\right] \tag{80c}$$

$$r_{59} = \left[\frac{4\alpha(r_1' + r_1)}{\theta} - \frac{9\alpha r_{29}}{4} + \frac{3\alpha r_{40}}{4} + \frac{11\beta r_{11}}{16}\right] \tag{80d}$$

$$r_{60} = \left[\frac{-9\alpha(r_4' + r_4)}{8\theta} - \frac{9\alpha r_{30}}{4} + \frac{3\alpha r_{41}}{4} + \frac{11\beta r_{12}}{16}\right] \tag{80e}$$

$$r_{61} = \left[\frac{-9\alpha(r_3' + r_3)}{8\theta} - \frac{9\alpha r_{31}}{4} + \frac{3\alpha r_{42}}{4} + \frac{11\beta r_{13}}{16}\right] \tag{80f}$$

$$r_{62} = \left[\frac{-9\alpha r_{32}}{4} + \frac{3\alpha r_{43}}{4} + \frac{11\beta r_{14}}{16}\right] \tag{80g}$$

$$r_{63} = \left[\frac{-9\alpha r_{33}}{4} + \frac{3\alpha r_{14}}{4} + \frac{11\beta r_{15}}{16}\right] \tag{80h}$$

$$r_{64} = \left[\frac{-9\alpha r_{34}}{4} + \frac{3\alpha r_{45}}{4} + \frac{11\beta r_{16}}{16}\right] \tag{80i}$$

$$r_{65} = \left[\frac{-9\alpha r_{35}}{4} + \frac{3\alpha r_{46}}{4} + \frac{11\beta r_{17}}{16}\right] \tag{80j}$$

$$r_{66} = \frac{3\alpha r_{47}}{4}, \quad r_{67} = \frac{3\alpha r_{48}}{4}, \quad r_{68} = \frac{3\alpha r_{49}}{4} \tag{80k}$$

$$r_{69} = \frac{3\alpha r_{51}}{4}, \quad r_{70} = \frac{3\alpha r_{51}}{4}, \quad r_{71} = \frac{3\alpha r_{52}}{4} \tag{80l}$$

$$r_{72} = \frac{3\alpha r_{47}}{4}, \quad r_{73} = \frac{3\alpha r_{48}}{4}, \quad r_{74} = \frac{3\alpha r_{49}}{4} \tag{80m}$$

$$r_{75} = \frac{3\alpha r_{56}}{4} \tag{80n}$$

$$r_{57}(0) = B^5\Omega_8, \quad \Omega_8 = \left[\alpha^2\left(\frac{28107}{384\theta^2} + \frac{3\Omega_2}{4}\right) + \frac{693\beta}{128\theta^2}\right] \tag{80o}$$

$$r_{58}(0) = B^5\Omega_9, \quad \Omega_9 = \left[\left(\frac{135}{7\theta^2} + \frac{475}{128\theta^2} + \frac{3\Omega_5}{4}\right) + \frac{693\beta}{128}\right] \tag{80p}$$

$$r_{59}(0) = \frac{-12\alpha B^5}{\theta}, \quad r_{60}(0) = B^5\Omega_{10}, \quad \Omega_{10} = \left[\alpha^2\left(\frac{3\Omega_4}{4} - \frac{1215}{128\theta^2}\right) - \frac{1375\beta}{256}\right] \tag{80q}$$

$$r_{61}(0) = \frac{9\alpha B^3}{16\theta}, \quad r_{62}(0) = B^5\Omega_{11}, \quad \Omega_{11} = \left(\frac{3\Omega_5}{4} - \frac{9}{62\theta^2} + \frac{11\beta}{64}\right) \tag{80r}$$

$$r_{63}(0) = 0, \quad r_{64}(0) = B^5\Omega_{12}, \quad \Omega_{12} = \left[\alpha^2\left(\frac{27}{518\theta^2} - \frac{3\Omega_6}{512\theta}\right) - \frac{11\beta}{128}\right] \tag{80s}$$

$$r_{65}(0) = 0, \quad r_{66}(0) = \frac{3\alpha^2 B^5\Omega_1}{4}, \quad r_{67}(0) = 0 \tag{80t}$$

$$r_{68}(0) = \frac{3\alpha^2 B^5\Omega_1}{4}, \quad r_{69}(0) = 0 \tag{80u}$$

$$r_{70}(0) = \frac{3\alpha^2 B^5\Omega_1}{16}, \quad r_{71}(0) = 0, \quad r_{72}(0) = \frac{3\alpha^2 B^5\Omega_1}{16} \tag{80v}$$

$$r_{73}(0) = 0, \quad r_{74}(0) = \frac{9\alpha^2 B^5\Omega_1}{4}, \quad r_{75}(0) = 0 \tag{80w}$$

The solution of (80a – z), using (78c) is

$$\begin{aligned} U_m^{(5)} = {} & \alpha_5(\tau)\cos\theta t + \gamma_5(\tau)\sin\theta t + \frac{r_{57}}{\theta^2} - \frac{1}{3\theta^2}(r_{58}\cos 2\theta t + r_{59}\sin 2\theta t) \\ & - \frac{1}{8\theta^2}(r_{60}\cos 3\theta t + r_{61}\sin 3\theta t) - \frac{1}{15\theta^2}(r_{62}\cos 4\theta t + r_{63}\sin 4\theta t) \\ & - \frac{1}{15\theta^2}(r_{64}\cos 5\theta t + r_{65}\sin 5\theta t) - \left(\frac{r_{66}\cos(\theta+\omega)t + r_{67}\sin(\theta+\omega)t}{\omega(\omega+2\theta)}\right) \\ & + \frac{(r_{68}\cos(\theta-\omega)t + r_{69}\sin(\theta-\omega)t)}{\omega(2\theta-\omega)} - \left(\frac{r_{70}\cos(2\theta+\omega)t + r_{71}\sin(2\theta+\omega)t}{(3\theta+\omega)(\theta+\omega)}\right) \\ & + \left(\frac{r_{72}\cos(\omega-2\theta)t + r_{73}\sin(\omega-2\theta)t}{(\omega-\theta)(3\theta-\omega)}\right) + \left(\frac{r_{74}\cos\omega t + r_{75}\sin\omega t}{(\theta+\omega)(\theta-\omega)}\right) \end{aligned} \tag{81a}$$

where,

$$\alpha_5(0) = \frac{B^5\Omega_{13}}{\theta^2} \tag{81b}$$

$$\Omega_{13} = \left[\frac{\Omega_9}{3} - \Omega_8 + \frac{\Omega_{10}}{8} + \frac{\Omega_{11}}{15} + \frac{\Omega_{12}}{24} + 3\alpha^2\left\{\frac{\Omega_1}{4\omega(\omega+2\theta)} - \frac{\Omega_1}{4\omega(2\theta-\omega)} + \frac{\Omega_1}{16(\theta+\omega)(3\theta+\omega)} - \frac{\Omega_1}{16(\omega-\theta)(3\theta-\omega)} - \frac{\Omega_1}{2(\theta+\omega)(\theta-\omega)}\right\}\right] \quad (81c)$$

$$\gamma_5(0) = \frac{1}{\theta}\left[\frac{2r_{59}}{3\theta} + \frac{2r_{61}}{8\theta} + \frac{2r_{63}}{15\theta} + \frac{5r_{65}}{15\theta} + \frac{(\theta+\omega)r_{67}}{\omega(2\theta+\omega)} - \frac{(\theta-\omega)r_{69}}{\omega(2\theta-\omega)} + \frac{(2\theta+\omega)r_{71}}{(3\theta+\omega)(\theta+\omega)} - \frac{(\omega-2\theta)r_{73}}{(\omega-\theta)(3\theta-\omega)} - \frac{\omega r_{75}}{(\theta+\omega)(\theta-\omega)}\right]\Big|_{\tau=0} + \frac{1}{\theta}\left[-\alpha_{12}'(0) + 3\alpha\left\{\frac{r_0'}{\theta^2} - \frac{r_1'}{3\theta^2} - \frac{r_3'}{8\theta^2}\right\}\right]\Big|_{\tau=0} \quad (81d)$$

Substituting into (78d) and ensuring a uniformly valid solution by equating to zero the coefficients of $cos\ \omega t$ and $sin\ \omega t$, the following are obtained respectively

$$\beta_3' + \beta_3 = -\frac{3\alpha r_{55}}{8\omega} \ and \ \alpha_3' + \alpha_3 = \frac{3\alpha r_{56}}{8\omega} \quad (82a, b)$$

On solving (82a, b), the result gives

$$\beta_3(\tau) = e^{-\tau}\left[\frac{-3\alpha}{8\omega}\int_0^\tau r_{55}e^s ds + \beta_3(0)\right] \quad (82c)$$

$$\alpha_3(\tau) = e^{-\tau}\left[\frac{3\alpha}{8\omega}\int_0^\tau e^s r_{56} ds\right] \quad (82d)$$

The remaining equation in the substitution into (78d) is

$$U_{3m,tt}^{(5)} + \omega^2 U_{3m}^{(5)} = r_{76} + r_{79}cos\theta t + r_{80}sin\theta t + r_{81}cos\ 2\theta t + r_{82}sin\ 2\theta t + r_{83}cos\ 3\theta t + r_{84}sin\ 3\theta t + r_{85}cos\ 4\theta t + r_{86}sin\ 4\theta t + r_{87}cos\ 5\theta t + r_{88}sin\ 5\theta t + r_{89}cos(\theta+\omega)t + r_{90}sin(\theta+\omega)t + r_{91}cos\ (\theta-\omega)t + r_{92}sin\ (\theta-\omega)t + r_{93}cos\ (2\theta+\omega)t + r_{94}sin\ (2\theta+\omega)t + r_{7295}cos\ (\omega-2\theta)t + r_{96}sin\ (\omega-2\theta)t \quad (83a)$$

$$U_{3m}^{(5)}(0,0) = 0, \quad U_{3m,t}^{(5)}(0,0) + U_{3m,\tau}^{(5)}(0,0) = 0 \quad (83b)$$

where,

$$r_{76} = \frac{3\alpha r_{25}}{4} - r_{36} - \frac{5\beta r_7}{16} \quad (83c)$$

$$r_{79} = \left[\frac{-2\theta(r_6' + r_6)\alpha}{\omega^2-\theta^2} + \frac{3\alpha r_{26}}{4} - \frac{3\alpha r_{37}}{4} - \frac{15\beta r_8}{16}\right] \quad (83d)$$

$$r_{80} = \left[\frac{2\theta(r_5' + r_5)\alpha}{\omega^2-\theta^2} + \frac{3\alpha r_{27}}{4} - \frac{3\alpha r_{38}}{4} - \frac{5\beta r_9}{16}\right] \quad (83e)$$

$$r_{81} = \left[\frac{4\theta(r_2' + r_2)\alpha}{\omega^2-\theta^2} + \frac{3\alpha r_{28}}{4} - \frac{3\alpha r_{39}}{4} - \frac{5\beta r_{10}}{16}\right] \quad (83f)$$

$$r_{82} = \left[\frac{4\theta(r_1' + r_1)\alpha}{\omega^2 - 4\theta^2} + \frac{3\alpha r_{29}}{4} - \frac{3\alpha r_{40}}{4} - \frac{5\beta r_{11}}{16}\right] \tag{83g}$$

$$r_{83} = \left[\frac{6\theta(r_4' + r_4)\alpha}{\omega^2 - 9\theta^2} + \frac{3\alpha r_{30}}{4} - \frac{3\alpha r_{41}}{4} - \frac{5\beta r_{12}}{16}\right] \tag{83h}$$

$$r_{84} = \left[\frac{6\theta(r_3' + r_3)\alpha}{\omega^2 - 9\theta^2} + \frac{3\alpha r_{31}}{4} - \frac{3\alpha r_{42}}{4} - \frac{5\beta r_{13}}{16}\right] \tag{83i}$$

$$r_{85} = \left[\frac{3\alpha r_{32}}{4} - \frac{3\alpha r_{43}}{4} - \frac{5\beta r_{14}}{16}\right] \tag{83j}$$

$$r_{86} = \left[\frac{3\alpha r_{33}}{4} - \frac{3\alpha r_{44}}{4} - \frac{5\beta r_{15}}{16}\right] \tag{83k}$$

$$r_{87} = \left[\frac{3\alpha r_{34}}{4} - \frac{3\alpha r_{45}}{4} - \frac{5\beta r_{16}}{16}\right] \tag{83l}$$

$$r_{88} = \left[\frac{3\alpha r_{35}}{4} - \frac{3\alpha r_{46}}{4} - \frac{5\beta r_{17}}{16}\right] \tag{83m}$$

$$r_{89} = \left[\frac{-3\alpha r_{47}}{4}\right], \qquad r_{90} = \left[-\frac{3\alpha r_{48}}{4}\right] \tag{83o}$$

$$r_{91} = \left[\frac{-3\alpha r_{49}}{4}\right], \qquad r_{92} = \left[-\frac{3\alpha r_{50}}{4}\right] \tag{83p}$$

$$r_{93} = \frac{-3\alpha r_{51}}{4}, \quad r_{94} = -\frac{3\alpha r_{52}}{4}, \quad r_{95} = -\frac{3\alpha r_{53}}{4}, \qquad r_{96} = -\frac{3\alpha r_{54}}{4} \tag{83q}$$

where,

$$r_{76}(0) = B^5\Omega_{14}, \quad \Omega_{14} = \left[\frac{3123\alpha}{128\theta^2} + \alpha\Omega_2 + \frac{315\beta}{128}\right] \tag{83r}$$

$$r_{79}(0) = B^5\Omega_{19}, \quad \Omega_{19} = \left[\frac{18198\alpha^2}{512\theta^2} = \frac{675\alpha^2}{16(\omega^2 - \theta^2)} - \frac{3\Omega_3}{4}\right] \tag{83s}$$

$$r_{80}(0) = B^3\Omega_{20}, \qquad \Omega_{20} = \frac{3\theta\alpha}{(\omega^2 - \theta^2)} \tag{83t}$$

$$r_{81}(0) = B^5\Omega_{21}, \qquad \Omega_{21} = -\left[\alpha^2\left(\frac{135}{2(\omega^2 - 4\theta^2)} - \frac{99}{8\theta^2} + \frac{3\Omega_4}{4}\right) + \frac{125\beta}{64}\right] \tag{83u}$$

$$r_{82}(0) = B^3\Omega_{22}, \qquad \Omega_{22} = -\left(\frac{12\theta\alpha}{\omega^2 - 4\theta^2}\right) \tag{83v}$$

$$r_{83}(0) = B^5\Omega_{23}, \quad \Omega_{23} = -\left[\alpha^2\left(\frac{405}{8(\omega^2 - 9\theta^2)B^2} + \frac{243}{512\theta^2} - \frac{3\Omega_4}{4}\right) + \frac{625\beta}{256}\right] \tag{83w}$$

$$r_{84}(0) = B^3\Omega_{24}, \qquad \Omega_{24} = -\left(\frac{3\theta\alpha}{\omega^2 - 9\theta^2}\right) \tag{83x}$$

$$r_{85}(0) = B^5\Omega_{25}, \quad \Omega_{25} = \left[\alpha^2\left(\frac{63}{128\theta^2} - \frac{3\Omega_5}{4}\right) - \frac{25\beta}{64}\right] \tag{83y}$$

$$r_{86}(0) = 0, \quad r_{87}(0) = B^5\Omega_{26}, \quad \Omega_{26} = \left[\alpha^2\left(\frac{9}{512\theta^2} + \frac{3\Omega_6}{4}\right) + \frac{5\beta}{128}\right] \tag{83z}$$

$$r_{88}(0) = 0, \quad r_{89}(0) = B^5\Omega_{27}, \quad \Omega_{27} = \frac{-3\alpha^2\Omega_1}{4} \tag{84a}$$

$$r_{90}(0) = 0, \quad r_{91}(0) = B^5\Omega_{28}, \quad \Omega_{28} = \frac{-3\alpha^2\Omega_1}{4}, \qquad r_{92}(0) = 0 \tag{84b}$$

$$r_{93}(0) = B^5\Omega_{29}, \quad \Omega_{29} = \frac{3\alpha^2\Omega_1}{16}, \qquad r_{94}(0) = 0 \tag{84c}$$

$$r_{95}(0) = 0 \quad B^5\Omega_{30}, \quad \Omega_{30} = \frac{3\alpha^2\Omega_1}{16}, \qquad r_{96}(0) = 0 \tag{84d}$$

In evaluating the above, the following facts are used

$$\alpha_3'(0) = -\alpha_3(0) = \alpha B^3\Omega_1; \quad \beta_3'(0) = \frac{9\alpha^2 B^5\Omega_1}{16\omega} \tag{84e}$$

The solution of (83a, b) is

$$\begin{aligned} U_{3m}^{(5)} = {} & \alpha_6(\tau)cos\,\omega t + \gamma_6(\tau)sin\,\omega t + \left(\frac{1}{\omega^2-\theta^2}\right)(r_{79}cos\theta t + r_{80}sin\theta t) \\ & + \left(\frac{1}{\omega^2-4\theta^2}\right)(r_{83}cos\,3\theta t + r_{84}sin\,3\theta t\,) + \left(\frac{1}{\omega^2-16\theta^2}\right)(r_{85}cos\,4\theta t + r_{86}sin\,4\theta t\,) \\ & + \left(\frac{1}{\omega^2-25\theta^2}\right)(r_{87}cos\,5\theta t + r_{88}sin\,5\theta t\,) \\ & + \left(\frac{1}{\theta(2\omega+\theta)}\right)(r_{89}cos(\theta+\omega)t + r_{90}sin(\theta+\omega)t) \\ & + \left(\frac{1}{\theta(2\omega+\theta)}\right)(r_{91}cos\,(\theta-\omega)t + r_{92}sin(\theta-\omega)t\,) \\ & - \left(\frac{1}{4\theta(\theta+\omega)}\right)(r_{93}cos\,(2\theta+\omega)t + r_{94}sin\,(2\theta+\omega)t\,) \\ & + \left(\frac{1}{4\theta(\omega-\theta)}\right)(r_{95}cos\,(\omega-2\theta)t + r_{96}sin\,(\omega-2\theta)t\,) \end{aligned} \tag{85a}$$

where,

$$\alpha_6(0) = B^5\Omega_{31} \tag{85b}$$

$$\begin{aligned} \Omega_{31} = -\Bigg[& \frac{\Omega_{31}}{\omega^2-\theta^2} + \frac{\Omega_{21}}{\omega^2-4\theta^2} + \frac{\Omega_{23}}{\omega^2-9\theta^2} + \frac{\Omega_{25}}{\omega^2-16\theta^2} + \frac{\Omega_{26}}{\omega^2-25\theta^2} - \frac{\Omega_{27}}{\theta(2\omega+\theta)} + \frac{\Omega_{28}}{\theta(2\omega-\theta)} \\ & - \frac{\Omega_{29}}{4\theta(\theta+\omega)} + \frac{\Omega_{30}}{4\theta(\omega-\theta)}\Bigg] \end{aligned} \tag{85c}$$

The determination of $\gamma_6(0)$ follows from using

$$U_{3m,t}^{(5)}(0,0) + U_{3m,\tau}^{(3)}(0,0) = 0 \tag{85d}$$

Here,

$$U_{3m,\tau}^{(3)}(0,0) = B^3\Omega_{32} \tag{85e}$$

$$\Omega_{32} = \left[\Omega_1 + \left\{-\frac{6}{\omega^2} + \frac{21}{4(\omega^2-\theta^2)} - \frac{6}{\omega^2-4\theta^2} + \frac{3}{4(\omega^2-9\theta^2)}\right\}\right] \quad (85f)$$

$$\therefore \quad \gamma_6(0) = B^3\Omega_{33} \quad (85g)$$

$$\Omega_{33} = -\frac{1}{\omega}\left[\frac{\theta\Omega_{20}}{\omega^2-\theta^2} + \frac{2\theta}{\omega^2-4\theta^2} + \frac{3\theta}{\omega^2-9\theta^2} + \Omega_{32}\right] \quad (85h)$$

Now, simplifying (78f), the following are obtained

$$U_{5m,tt}^{(5)} + \varphi^2 U_{5m}^{(5)} = \frac{\beta}{16}\left(U_m^{(1)}\right)^5$$

$$= \frac{\beta}{16}[r_7 + r_8 cos\theta t + r_9 sin\theta t + r_{10} cos\, 2\theta t + r_{11} sin\, 2\theta t + r_{12} cos\, 3\theta t + r_{13} sin\, 3\theta t$$
$$+ r_{14} cos\, 4\theta t + r_{15} sin\, 4\theta t + r_{87}\, cos\, 5\theta t + r_{17} sin\, 5\theta t\ ] \quad (86a)$$
$$U_{5m}^{(5)}(0,0) = U_{5m,t}^{(5)} = 0 \quad (86b)$$

The solution of (86a, b) is

$$U_{5m}^{(5)} = \alpha_7 cos\,\varphi t + \gamma_7 sin\,\varphi t$$
$$+ \frac{\beta}{16}\left[\frac{r_7}{\varphi^2} + \left(\frac{1}{\varphi^2-\theta^2}\right)(r_8 cos\theta t + r_9 sin\theta t) + \left(\frac{1}{\varphi^2-4\theta^2}\right)(r_{10} cos\, 2\theta t + r_{11} sin\, 2\theta t\ )\right.$$
$$+ \left(\frac{1}{\varphi^2-9\theta^2}\right)(r_{12} cos\, 3\theta t + r_{13} sin\, 3\theta t\ ) + \left(\frac{1}{\varphi^2-16\theta^2}\right)(r_{14} cos\, 4\theta t + r_{15} sin\, 4\theta t\ )$$
$$\left. + \left(\frac{1}{\varphi^2-25\theta^2}\right)(r_{14} cos\, 5\theta t + r_{15} sin\, 5\theta t\ )\right] \quad (87a)$$

$$\alpha_7(0) = B^5\Omega_{30}^{(1)}, \quad \gamma_7(0) = 0 \quad (87b)$$

where,

$$\Omega_{30}^{(1)} = -\frac{\beta}{16}\left[\frac{63}{8\varphi^2} - \frac{105}{8(\varphi^2-\theta^2)} + \frac{25}{4(\varphi^2-4\theta^2)} - \frac{125}{16(\varphi^2-9\theta^2)} + \frac{5}{4(\varphi^2-16\theta^2)}\right.$$
$$\left. - \frac{1}{8(\varphi^2-25\theta^2)}\right] \quad (87c)$$

The determination of $U_m^{(4)}$ and $U_{3m}^{(4)}$ in full will automatically depend on $U_m^{(2)}$ which vanishes as in (73c). Hence, it can be concluded that

$$U^{(4)} = U_m^{(4)} = U_{3m}^{(4)} \equiv 0$$

Thus, the buckling mode takes the form

$$U(x,t,\tau,\epsilon) = \epsilon U_m^{(1)} sinmx + \epsilon^3\left(U_m^{(3)} sinmx + U_{3m}^{(3)} sin3mx\right)$$
$$+ \epsilon^5\left(U_m^{(5)} sinmx + U_{3m}^{(5)} sin3mx + U_{5m}^{(5)} sin5mx\right) + \cdots \quad (88)$$

7 Dynamic Buckling Load

According to Amazigo [18], the dynamic buckling load λ_D is obtained through the maximization

$$\frac{d\lambda}{dU_a} = 0 \quad (89)$$

where U_a is the maximum displacement . The dynamic buckling load is the largest value of the load parameter for the solution to be bounded. The onus on us now is to first determine U_a subsequent upon which (89) shall be invoked to determine the value of λ_D. The conditions for the maximum U_a are

$$U_{,x} = 0, \qquad U_{,t} + \epsilon^2 U_{,\tau} = 0 \quad (90a, b)$$

From (89), it can be seen that the least nontrivial value of $x = x_a$, for $U_x = 0$ is

$x_a = \frac{\pi}{2m}$.

Thus, from (89), the following is obtained

$$U(x_a, t, \tau, \epsilon) = \epsilon U_m^{(1)} + \epsilon^3\left(U_m^{(3)} - U_{3m}^{(3)}\right) + \epsilon^5\left(U_m^{(5)} - U_{3m}^{(5)} + U_{5m}^{(5)}\right) + \cdots \quad (91)$$

Let the values of t and τ at maximum displacement be t_a and τ_a respectively and

$$t_a = t_0^{(1)} + \epsilon^2 t_2^{(1)} + \epsilon^4 t_4^{(1)} + \cdots$$
$$\tau_a = \epsilon^2 t_a = \epsilon^2\left(t_0^{(1)} + \epsilon^2 t_2^{(1)} + \epsilon^4 t_4^{(1)} + \cdots\right)$$

The results will be given in two separate levels of approximations, first, by taking all the buckling modes in the shape of imperfection and secondly, by taking the buckling modes in the combined shapes of $sin\ mx$ and $sin\ 3mx$.

7.1 Dynamic buckling load for modes in the shape of imperfection

In this case, (91) becomes

$$U = \epsilon U_m^{(1)} + \epsilon^3 U_m^{(1)} + \epsilon^5 U_m^{(1)} + \cdots \quad (92a)$$

By substituting (92a) into (90b) and equating coefficients of ϵ, the following are obtained

$$O(\epsilon):\ U_{m,t}^{(1)}\left(t_0^{(1)}, 0\right) = 0 \quad (92b)$$

$$O(\epsilon^2):\ t_2^{(1)} U_{m,tt}^{(1)} + U_{m,\tau}^{(1)} = 0 \quad (92c)$$

etc.

where, (92b, c) are evaluated at $\left(t_0^{(1)}, 0\right)$. From (92b), it is got that, $t_0^{(1)} = \frac{\pi}{\theta}$ and from (92c),

$$t_2^{(1)} = \frac{1}{\theta^2}$$

After evaluating (92a) at maximum value, the non – vanishing terms become

$$U_a = \Big[\epsilon U_m^{(1)} + \epsilon^3\left(t_0^{(1)} U_{m,\tau}^{(1)} + U_m^{(3)}\right) + \epsilon^5\left(t_2^{(1)} U_{m,\tau}^{(1)} + \frac{1}{2} {t_2^{(1)}}^2 U_{m,tt}^{(1)} + \frac{1}{2} {t_0^{(1)}}^2 U_{m,\tau\tau}^{(1)} + U_{m,\tau}^{(3)} + U_m^{(5)}\right)\Big] \Big|_{\left(t_0^{(1)}, 0\right)} + \cdots \quad (93a)$$

On simplifying (93a), the following is obtained

$$U_a = g_1\epsilon + g_2\epsilon^3 + g_3\epsilon^5 + \cdots \qquad (93b)$$

where,

$$g_1 = 2B, \qquad g_2 = \frac{-18B^3 Q_{40}}{\theta^2}, \qquad Q_{40} = \left(1 + \frac{\pi\theta}{8\alpha B^2}\right) \qquad (93c)$$

$$g_3 = \frac{2B^5 \Omega_{38} Q_{41}}{\theta^2}, \qquad (93d)$$

$$Q_{41} = \left[1 + \left(\frac{\theta^2}{25\Omega_{38}B^5}\right)\left\{\left(\frac{\pi}{\theta}\right)^2\left(\frac{B - 2025\alpha^2 B^5}{128\theta^2}\right) + \left(\frac{\pi}{\theta}\right)\left(\frac{27\alpha B^3}{\theta^2}\right)\right\}\right] \qquad (94a)$$

and

$$\Omega_{38} = \left[\Omega_8 - \frac{\Omega_9}{3} - \frac{\Omega_1}{5} - \frac{\theta^2\alpha^2}{2}\left\{\left\{\frac{3\Omega_1}{4\omega(w+2\theta)} + \frac{3\Omega_1}{4\omega(2\theta-w)} + 3\Omega_1\left\{\frac{1+cos\left(\frac{\pi\omega}{\theta}\right)}{16(3\theta+w)(\theta+w)}\right\} - \frac{3\Omega_1\left\{1+cos\left(\frac{\pi\omega}{\theta}\right)\right\}}{16(\omega-\theta)(3\theta-w)} - \right.\right.\right.$$

$$9\Omega 11+cos\,\pi\omega\theta\ 16\theta+\omega\theta-w \qquad (94b)$$

Since the series (93b) does not converge when $U_a > U_{aD}$, where U_{aD} is the critical displacement at dynamic buckling, as in [18], (93b) is reversed in the form

$$\epsilon = l_1 U_a + l_2 U_a^3 + l_3 U_a^5 + \cdots \qquad (95a)$$

By substituting in (95a) for U_a from (93b), and equating the coefficients of powers of ϵ, the following are obtained

$$l_1 = \frac{1}{g_1} = \frac{1}{2B}, \quad l_2 = -\frac{g_2}{g_1^4} = \frac{9\alpha Q_{40}}{8B}$$

$$l_3 = \frac{3g_2^2 - g_1 g_3}{g_1^7} = -\frac{g_1 g_3}{g_1^7}\left(1 - \frac{3g_2^2}{g_1 g_3}\right) = -\frac{g_3}{g_1^6}\left(1 - \frac{3g_2^2}{g_1 g_3}\right) = -\frac{\Omega_{38}Q_{41}Q_{42}}{16B\theta^2}$$

where,

$$Q_{42} = \left(1 - \frac{243\alpha^2 Q_{40}^2}{\Omega_{38}Q_{41}}\right)$$

In order to determine the dynamic buckling load λ_D, the maximization (89) is now carried out using (95a) to get

$$l_1 + 3l_2 U_{aD}^2 + 5l_3 U_{aD}^4 = 0 \qquad (95b)$$

where, U_{aD} = $U_a(\lambda_D)$.This yields

$$U_{aD}^2 = \frac{27}{5}\left(\frac{\left(-\frac{\theta^2}{\alpha}\right)}{\Omega_{38}Q_{41}Q_{42}}\right)\left[-1 - \sqrt{1 + \frac{2\Omega_{38}Q_{41}Q_{42}}{(9\alpha\theta Q_{40})^2}}\right]$$

$$\therefore \quad U_{aD} = \sqrt{\frac{27}{5}}\sqrt{\frac{\left(-\frac{\theta^2}{\alpha}\right)}{\Omega_{38}Q_{41}Q_{42}}}\left[-1-\sqrt{1+\frac{2\Omega_{38}Q_{41}Q_{42}}{(9\alpha\theta Q_{40})^2}}\right]^{\frac{1}{2}} \tag{95c}$$

To determine the dynamic buckling load λ_D, (95a) is evaluated at dynamic buckling condition where $U_a = U_{aD}$. Next, (95a) is multiplied by 5. By making $5l_3U_{aD}^4$ the subject in (95b) and substituting same in (95a), (as multiplied by 5) and simplifying, the following is obtained

$$5\epsilon = \frac{2U_{aD}}{B}\left[1+\frac{9\alpha Q_{40}U_{aD}^2}{8}\right] \tag{95d}$$

On simplifying (95d), the following is obtained

$$5m^2\underline{a}_m\epsilon\lambda_D = (m^4 - 2m^2\lambda_D + 1)U_{aD}\left[1+\frac{9\alpha Q_{40}U_{aD}^2}{8}\right] \tag{96}$$

It must be stressed that equation (96) is evaluated at $\lambda = \lambda_D$ and it is implicit in λ_D.

7.2 Dynamic Buckling Load in the Shape of Both $sin\, mx$ and $sin\, 3mx$

In this case, (91) becomes

$$U(x_a, t, \tau, \epsilon) = \epsilon U_m^{(1)} + \epsilon^3\left(U_m^{(3)} - U_{3m}^{(3)}\right) + \epsilon^5\left(U_m^{(5)} - U_{3m}^{(5)}\right) + \cdots \tag{97a}$$

Let the values of t and τ at maximum displacement of (97a) be t_c and τ_c respectively, where

$$t_c = t_0^{(2)} + \epsilon^2 t_2^{(2)} + \epsilon^4 t_4^{(2)} + \cdots \tag{97b}$$

$$\tau_c = \epsilon^2 t_c = \epsilon^2\left(t_0^{(2)} + \epsilon^2 t_2^{(2)} + \epsilon^4 t_4^{(2)} + \cdots\right) \tag{97c}$$

By substituting (97b, c) into (90b) and expanding as usual (similar to operations leading to (92b, c)), the non – vanishing terms become

$$O(\epsilon):\ U_{m,t}^{(1)} = 0 \tag{98a}$$

$$O(\epsilon^3):\ t_2^{(2)}U_{m,tt}^{(1)} - U_{3m,t}^{(3)} + U_{m,\tau}^{(1)} = 0 \tag{98b}$$

From (98a), the following is obtained

$$t_0^{(2)} = \frac{\pi}{\theta} \tag{98c}$$

and from (98b), the following are obtained

$$t_2^{(2)} = \frac{1}{\theta^2}(1 - B^2\Omega_{34}) \tag{98d}$$

$$\Omega_{34} = \alpha\omega\left[\frac{4}{\omega^2} - \frac{15}{4(\omega^2-\theta^2)} + \frac{3}{(\omega^2-4\theta^2)} - \frac{1}{4(\omega^2-9\theta^2)}\right] sin\, sin\, \left(\frac{\omega\pi}{\theta}\right)$$

Now, determining the maximum U_c of (97a), using (97b, c) and (98c, d), the following non – vanishing terms are obtained

$$U_c = \epsilon U_m^{(1)} + \epsilon^3\left(t_0^{(2)}U_{m,\tau}^{(1)} + U_m^{(3)} - U_{3m}^{(3)}\right) + \epsilon^5\left[t_2^{(2)}U_{m,\tau}^{(1)} + \frac{1}{2}t_2^{(2)^2}U_{m,tt}^{(1)} + t_0^{(2)^2}U_{m,\tau\tau}^{(1)} + t_2^{(2)}\left(U_m^{(3)} - U_{3m}^{(3)}\right)_{,t} + t_0^{(2)}\left(U_m^{(3)} - U_{3m}^{(3)}\right)_{,\tau} + \left(U_m^{(5)} - U_{3m}^{(5)}\right)\right] + \cdots \quad (99a)$$

where,

$$U_{3m}^{(3)}\left(t_0^{(2)},0\right) = \alpha B^3 \Omega_{35} \quad (99b)$$

$$\Omega_{35} = \left[4\left\{\frac{1-\cos\left(\frac{\omega\pi}{\theta}\right)}{\omega^2}\right\} - \frac{15\left\{1-\cos\left(\frac{\omega\pi}{\theta}\right)\right\}}{4(\omega^2-\theta^2)} + \frac{3\left\{1-\cos\left(\frac{\omega\pi}{\theta}\right)\right\}}{(\omega^2-4\theta^2)} - \frac{15\left\{1-\cos\left(\frac{\omega\pi}{\theta}\right)\right\}}{4(\omega^2-9\theta^2)}\right] \quad (99c)$$

$$U_{3m,t}^{(3)}\left(t_0^{(2)},0\right) = \alpha\omega B^5 \Omega_{36} \quad (100a)$$

$$\Omega_{36} = \left[\frac{4}{\omega^2} - \frac{15}{4(\omega^2-\theta^2)} + \frac{3}{(\omega^2-4\theta^2)} - \frac{15}{4(\omega^2-9\theta^2)}\right]\cos\left(\frac{\omega\pi}{\theta}\right)$$

$$U_{m,\tau}^{(3)}\left(t_0^{(2)},0\right) = \frac{27\alpha B^3}{\theta^2}, \qquad U_{3m,\tau}^{(3)}\left(t_0^{(2)},0\right) = \alpha B^3 \Omega_{37}$$

$$\Omega_{37} = \left[\Omega_1\cos\left(\frac{\omega\pi}{\theta}\right) + 9\alpha\Omega_1 B^3 \sin\left(\frac{\omega\pi}{\theta}\right) - 3\left\{\frac{2}{\omega^2} + \frac{7}{4(\omega^2-\theta^2)} + \frac{2}{(\omega^2-4\theta^2)} + \frac{1}{4(\omega^2-9\theta^2)}\right\}\right] \quad (100b)$$

$$U_m^{(5)}\left(t_0^{(2)},0\right) = \frac{2B^5}{\theta^2}\Omega_{38} \quad (101a)$$

$$\Omega_{38} = \Omega_8 - \frac{\Omega_9}{3} - \frac{\Omega_1}{5} - \frac{\theta^2\alpha^2}{2}\left\{\left\{\frac{3\Omega_1}{4\omega(\omega+2\theta)} + \frac{3\Omega_1}{4\omega(2\theta-\omega)} + \frac{3\Omega_1\left\{1+\cos\left(\frac{\omega\pi}{\theta}\right)\right\}}{16(3\theta+\omega)(\theta+\omega)} - \frac{3\Omega_1\left\{1+\cos\left(\frac{\omega\pi}{\theta}\right)\right\}}{16(3\theta+\omega)(\theta+\omega)} - \frac{9\Omega_1\left\{1+\cos\left(\frac{\omega\pi}{\theta}\right)\right\}}{16(\theta+\omega)(\theta-\omega)}\right\}\right\} \quad (101b)$$

$$U_{3m}^{(5)}\left(t_0^{(2)},0\right) = B^5\Omega_{39} \quad (102a)$$

$$\Omega_{39} = \left[\Omega_{39}cos\left(\frac{\omega\pi}{\theta}\right) + \frac{\Omega_{33}}{B^2}sin\left(\frac{\omega\pi}{\theta}\right) - \frac{\Omega_{19}}{\omega^2-\theta^2} + \frac{\Omega_{21}}{(\omega^2-4\theta^2)} - \frac{\Omega_{23}}{4(\omega^2-9\theta^2)} + \frac{\Omega_{25}}{4(\omega^2-16\theta^2)} - \frac{\Omega_{26}}{4(\omega^2-25\theta^2)} + \frac{\Omega_{27}cos\left(\frac{\omega\pi}{\theta}\right)}{\theta(2\omega+\theta)} + \frac{\Omega_{28}cos\left(\frac{\omega\pi}{\theta}\right)}{\theta(2\omega-\theta)} - \frac{\Omega_{29}cos\left(\frac{\omega\pi}{\theta}\right)}{4\theta(\theta+\omega)} + \frac{\Omega_{30}cos\left(\frac{\omega\pi}{\theta}\right)}{4\theta(\omega-\theta)}\right] \quad (102b)$$

The maximum U_c of (99a) is now determined using (98c, d) and (97b, c), to get

$$U_c = 2B\epsilon + \epsilon^3\left[-\frac{B\pi}{\theta} - \alpha B^3\left(\frac{18}{\theta^2} + \Omega_{35}\right)\right] + \epsilon^5\left[-B\left(t_2^{(2)} + \frac{\theta^2 {t_2^{(2)}}^2}{2}\right) + \frac{{t_0^{(2)}}^2}{2}(B - 2025\alpha^2B^5) - t_2^{(2)}\omega\alpha B^5\Omega_{36} + B^3t_0^{(2)}\alpha\left(\frac{27}{\theta^2} - \Omega_{37}\right) + \frac{2B^5\Omega_{38}}{\theta^2} + B^5\Omega_3\right] \quad (103a)$$

Further simplification of (103a) gives

$$U_c = \epsilon q_1^{(2)} + \epsilon^3 q_2^{(2)} + \epsilon^5 q_3^{(2)} \quad (103b)$$

$$q_1^{(2)} = 2B, \quad q_2^{(2)} = \frac{-18\alpha B^3}{\theta^2}Q_{56} \quad (103c)$$

$$Q_{56} = \left[1 + \frac{\theta^2}{18\alpha B^3}\left(\frac{B\pi}{\theta\alpha} + \alpha\Omega_{35}\right)\right]$$

$$q_3^{(2)} = \frac{2B^5\Omega_{38}Q_{57}}{\theta^2} \quad (103d)$$

$$Q_{57} = \left[1 + \frac{\theta^2}{2B^5\Omega_{38}}\left\{B\Omega_{39} - B\left(t_2^{(2)} + \frac{1}{2}{t_2^{(2)}}^2\theta\right) + \frac{{t_0^{(2)}}^2}{2}(B - 2025\alpha^2B^5) - t_2^{(2)}\alpha\omega B^5\Omega_{36} + t_0^{(2)}\left(\frac{27\alpha B^3}{\theta^2} - \alpha B^5\Omega_{37}\right)\right\}\right] \quad (103e)$$

The series (103b) is now reversed by letting

$$\epsilon = h_1U_c + h_2U_c^3 + h_3U_c^5 \quad (104)$$

where,

$$h_1 = \frac{1}{q_1^{(2)}} = \frac{1}{2B}, \quad h_2 = \frac{q_2^{(2)}}{{q_1^{(2)}}^4} = \frac{9\alpha Q_{56}}{8B\theta^2}$$

$$h_3 = \frac{{q_2^{(2)}}^2 - q_1^{(2)}q_3^{(2)}}{{q_1^{(2)}}^7} = -\frac{q_2^{(2)}}{{q_1^{(2)}}^6}\left(1 - \frac{3q_2^{(2)}}{q_1^{(2)}q_3^{(2)}}\right) = \frac{-\Omega_{38}9\alpha Q_{57}Q_{58}}{32B\theta^2},$$

$$Q_{58} = \left[1 - \frac{243\alpha^2Q_{56}^2}{\theta^2\Omega_{38}Q_{57}}\right]$$

The maximization $\frac{d\lambda}{dU_c} = 0$ to obtain the dynamic buckling load λ_D yields, through (104),

$$5h_3U_{cD}^4 + 3h_2U_{cD}^2 + h_1 = 0 \qquad (105a)$$

where, $U_{cD} = U_c(\lambda_D)$ and is the value of the displacement at dynamic buckling stage. This yields

$$U_{cD}^2 = \frac{-3h_2 \pm \sqrt{9h_2^2 - 20h_1h_3}}{10h_3}$$
$$= \left(\frac{3h_2}{10h_3}\right)\left[-1 \pm \sqrt{1 - \frac{20h_1h_3}{9h_2^2}}\right] \qquad (105b)$$
$$= \frac{54}{5}\left(\frac{Q_{56}(-\alpha)}{\Omega_{38}Q_{57}Q_{58}}\right)\left[-1 \pm \sqrt{1 + \frac{20\theta^2\Omega_{38}Q_{57}Q_{58}}{729(\alpha Q_{56})^2}}\right]$$

Therefore,

$$U_{cD} = \left(\frac{54}{5}\right)^{\frac{1}{2}}\left(\frac{Q_{56}(-\alpha)}{\Omega_{38}Q_{57}Q_{58}}\right)^{\frac{1}{2}}\left[-1 \pm \left\{1 + \frac{20\theta^2\Omega_{38}Q_{57}Q_{58}}{729(\alpha Q_{56})^2}\right\}^{\frac{1}{2}}\right]^{\frac{1}{2}} \qquad (105c)$$

The dynamic load λ_D in this case is obtained by first multiplying (104) by 5 to get

$$5\epsilon = U_c[5h_1 + 5h_2U_c^2 + 5h_3U_c^4] \qquad (106a)$$

From (105a), $5h_3U_c^4$ is made the subject and substituting same in (106a) and simplifying, the following are obtained

$$5m^2\underline{a}_m\epsilon\lambda_D = (m^4 - 2m^2\lambda_D + 1)U_c\left[1 + \frac{9\alpha Q_{56}U_c^2}{8\theta^2}\right] \qquad (106b)$$

Equation (106b) gives an implicit equation for determining the dynamic buckling load λ_D in the case of buckling modes that are in the shapes of sinmx and sin3mx.

8 Discussion of Results

The results (41), (48b), (96) and (106b) are all implicit in the corresponding load parameters and are valid provided the parameters of asymptotic expansions are really small compared to unity. Using equations (41) and (96) on one hand, and (48b) and (106b) on the other hand, we can easily determine the mathematical relationship between the dynamic buckling load λ_D and the corresponding static buckling load λ_S. These are respectively given as

$$\left(\frac{\lambda_D}{\lambda_S}\right) = \frac{1}{4}\left(\frac{m^4 - 2m^2\lambda_D + 1}{m^4 - 2m^2\lambda_S + 1}\right)\frac{U_{aD}}{w_a}\left[\frac{1 + \frac{9\alpha Q_{40}U_{aD}^2}{8}}{1 - \frac{3\alpha Q_{51}w_a^2}{8\theta^2}}\right] \qquad (107)$$

$$\left(\frac{\lambda_D}{\lambda_S}\right) = \frac{1}{4}\left(\frac{m^4 - 2m^2\lambda_D + 1}{m^4 - 2m^2\lambda_S + 1}\right)\frac{U_c}{w_{ac}}\left[\frac{1 + \frac{9\alpha Q_{56}U_c^2}{8}}{1 + \frac{3\alpha Q_{51}w_{ac}^2}{8\theta^2}}\right] \qquad (108)$$

9 Main Results and Their Significance

In this analysis, the static buckling load of the structure for the case in which the buckling mode is strictly in the case of imperfection has been determined. The result is

$$5\epsilon = \frac{4w_a(m^4 - 2m^2\lambda_S + 1)}{m^2\underline{a}_m\lambda_S}\left[1 - \frac{3\alpha w_a^2(\lambda_S)}{8\theta^2}\right] \tag{109}$$

Also, the static buckling load for the case where the buckling mode is partly in the shape of imperfection and partly in the shape of $sin3mx$ is determined. The result is

$$5m^2\underline{a}_m\epsilon\lambda_S = 4w_{ac}(m^4 - 2m^2\lambda_S + 1)\left[1 + \frac{3\alpha w_{ac}^2 Q_{51}}{8\theta^2}\right] \tag{110}$$

In the same way, the dynamic buckling load of the structure for the case in which the buckling mode is strictly in the case of imperfection has been determined. The result is

$$5m^2\underline{a}_m\epsilon\lambda_D = (m^4 - 2m^2\lambda_D + 1)U_{aD}\left[1 + \frac{9\alpha Q_{40}U_{aD}^2}{8}\right] \tag{111}$$

Finally, the dynamic buckling load of the structure for the case in which the buckling mode is partly in the shape of imperfection and partly in the shape of $sin3mx$ is

$$5m^2\underline{a}_m\epsilon\lambda_D = (m^4 - 2m^2\lambda_D + 1)U_c\left[1 + \frac{9\alpha Q_{56}U_c^2}{8\theta^2}\right] \tag{112}$$

Using equations (109) and (111), the dynamic buckling load is related to the static buckling load in the case where the buckling mode is strictly in the case of imperfection, as

$$\left(\frac{\lambda_D}{\lambda_S}\right) = \frac{1}{4}\left(\frac{m^4 - 2m^2\lambda_D + 1}{m^4 - 2m^2\lambda_S + 1}\right)\frac{U_{aD}}{w_a}\left[\frac{1 + \frac{9\alpha Q_{40}U_{aD}^2}{8}}{1 - \frac{3\alpha Q_{51}w_a^2}{8\theta^2}}\right] \tag{113}$$

In the same way, using equations (110) and (112), the dynamic buckling load is related to the static buckling load in the case where the buckling mode is partly in the shape of imperfection and partly in the shape of $sin3mx$. The result is

$$\left(\frac{\lambda_D}{\lambda_S}\right) = \frac{1}{4}\left(\frac{m^4 - 2m^2\lambda_D + 1}{m^4 - 2m^2\lambda_S + 1}\right)\frac{U_c}{w_{ac}}\left[\frac{1 + \frac{9\alpha Q_{56}U_c^2}{8}}{1 + \frac{3\alpha Q_{51}w_{ac}^2}{8\theta^2}}\right] \tag{114}$$

The significance of $(113) and$ (114) is that, given any of $\lambda_D$or λ_S, the other value can be found without the labour of repeating the same asymptotic and perturbation procedures for the same imperfection parameter.

10 Numerical Results and Graphical Plots

Numerical values and graphical plots of the results were obtained using Q-Basic codes and the results are hereby presented in Tables 1, 2, 3, Figs. 1, 2 and 3 below.

Table 1: Relationship between Imperfection Parameter $\underline{a}_1\epsilon$ and Static Buckling Load λ_S for $m = 1, \alpha = 1, \beta = 1$ and $\underline{a}_1 = 0.01$, using modes in the shape of imperfection, as in eqn (41)

Imperfection Parameter $\underline{a}_1\epsilon$	Static Buckling Load λ_S
0.01	0.593423
0.02	0.593421
0.03	0.59342
0.04	0.593419
0.05	0.593418
0.06	0.593417
0.07	0.593416
0.08	0.593415
0.09	0.593414
0.1	0.593413

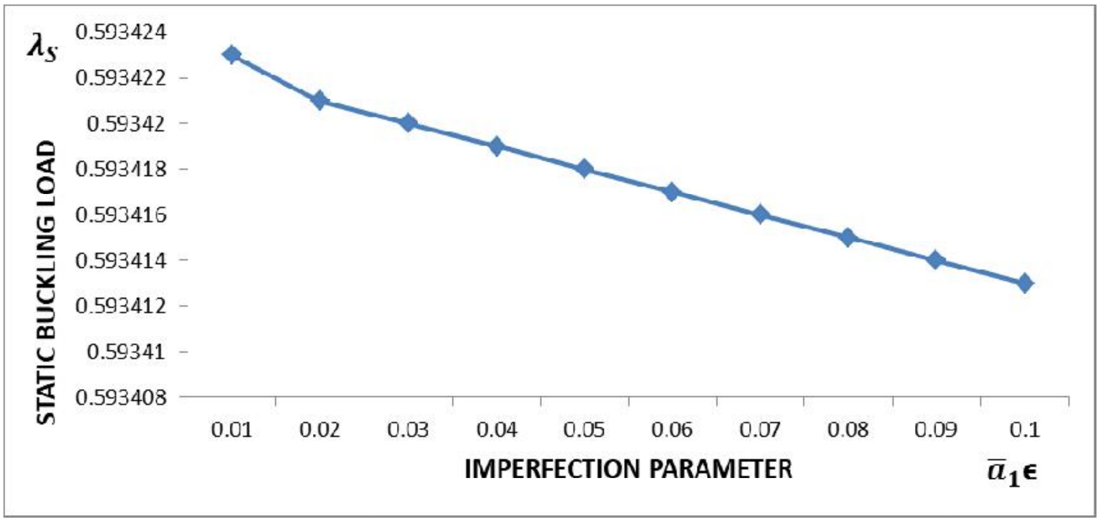

Fig. 1. Graphical plot showing the relationship between Imperfection Parameter $\underline{a}_1\epsilon$ and Static Buckling Load λ_S for $m = 1, \alpha = 1, \beta = 1$ and $\underline{a}_1 = 0.01$, using modes in the shape of imperfection

Table 2. Relationship between Imperfection Parameter $\underline{a}_1\epsilon$ and Static Buckling Load λ_S for $m = 1, \alpha = 1, \beta = 1$ and $\underline{a}_1 = 0.01$, in the case of modes in the shapes of combined $sin\, sin\, mx$ and $sin\, sin\, 3mx$, as in eqn (48b)

Imperfection Parameter $\underline{a}_1\epsilon$	Static Buckling Load λ_S
0.01	0.312078
0.02	0.281438
0.03	0.258314
0.04	0.239293
0.05	0.222829
0.06	0.208059
0.07	0.194414
0.08	0.181451
0.09	0.168739
0.1	0.155691

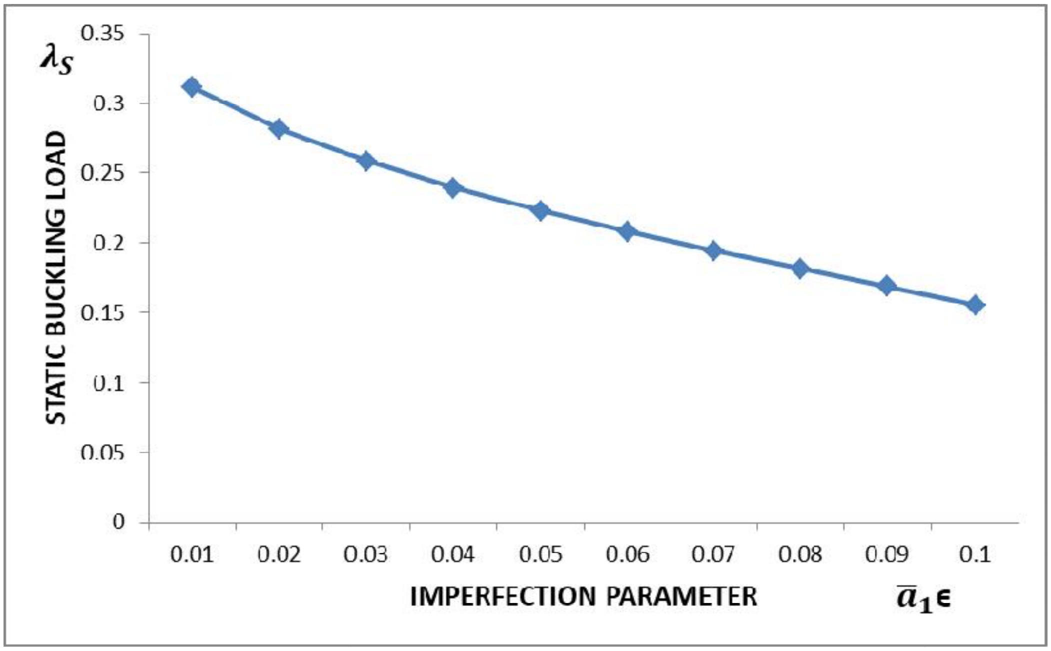

Fig. 2. Relationship between Imperfection Parameter $\underline{a}_1\epsilon$ and Static Buckling Load λ_S for $m = 1, \alpha = 1, \beta = 1$ and $\underline{a}_1 = 0.01$, in the case of modes in the shapes of combined $sin\ sin\ mx$ and $sin\ sin\ 3mx$

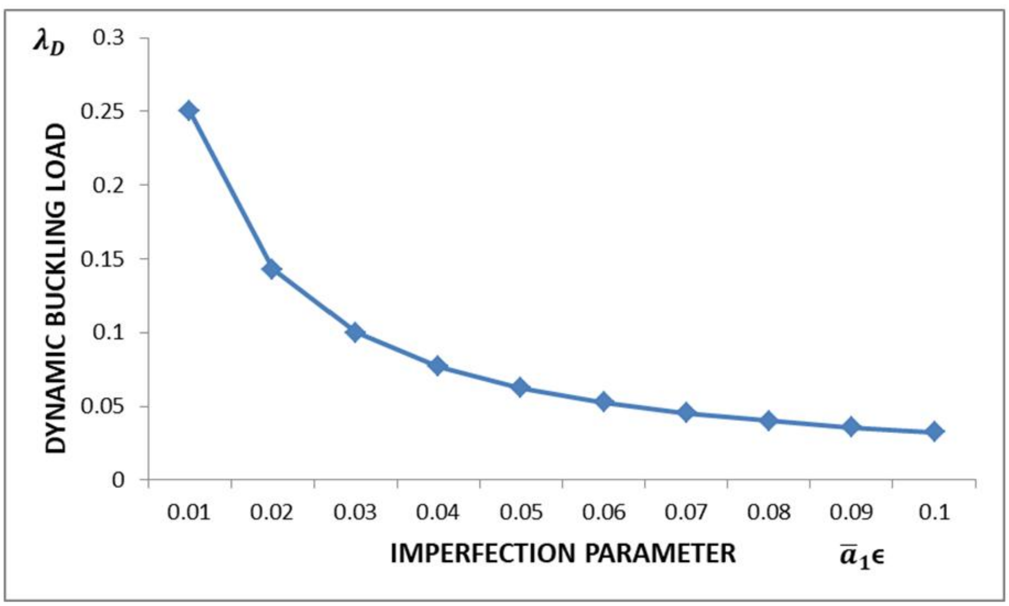

Fig. 3. Graphical plot showing the relationship between Imperfection Parameter $\underline{a}_1\epsilon$ and Dynamic Buckling Load λ_D for $m = 1, \alpha = 1, \beta = 1$ and $\underline{a}_1 = 0.01$, using modes in the shape of imperfection

Table 3. Relationship between Imperfection Parameter $\underline{a}_1\epsilon$ and Dynamic Buckling Load λ_D for $m = 1, \alpha = 1, \beta = 1$ and $\underline{a}_1 = 0.01$, using modes in the shape of imperfection, as in eqn (96)

Imperfection Parameter $\underline{a}_1\epsilon$	Dynamic Buckling Load λ_D
0.01	0.250018
0.02	0.142869
0.03	0.100011
0.04	0.076929
0.05	0.062461
0.06	0.052657
0.07	0.045459
0.08	0.040004
0.09	0.035718
0.1	0.032261

11 Conclusion

In this paper, the static and dynamic buckling loads of a viscously damped column lying on a cubic – quintic nonlinear elastic foundation stressed by a step load (in the dynamic loading case) have been dtermined. All results are asymptotic and implicit in the load parameters. The implicit nature of results notwithstanding, this work has been able to relate the dynamic buckling load λ_D to its corresponding static equivalent λ_S. This shows that if one of these buckling loads is known, then the other can be obtained easily. Specifically, the following are obtained from the graphical plots:

(a) This static and dynamic buckling loads decrease with increased imperfection,
(b) The static buckling load, for the case of buckling modes in the case of sinmx, appears to be higher than the corresponding static buckling of the case of buckling modes that are in the combined shapes of sinmx and sin3mx,
(c) The dynamic load is significantly lower than the corresponding static buckling load for the same imperfection.

Competing Interests

Authors have declared that no competing interests exist.

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