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A Power Gompertz Distribution: Model, Properties and Application to Bladder Cancer Data

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

This paper uses a power transformation approach to introduce a three-parameter probability distribution which gives another extension of the Gompertz distribution known as "Power Gompertz distribution". The statistical features of the power Gompertz distribution are systematically derived and studied appropriately. The three parameters of the new model are being estimated using the method of maximum likelihood estimation. The proposed distribution has also been compared to the Gompertz distribution using a real life dataset and the result shows that the Power Gompertz distribution has better performance than the Gompertz distribution and hence will be more useful and effective if applied in some real life situations especially survival analysis and cure fraction modeling just like the conventional Gompertz distribution.

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AMS Classication: 60E05, 62FXX, 62F10, 62G05, 90B25.

1 Introduction

The Gompertz distribution is both skewed to the right and to the left. It is a generalization of the exponential distribution and is commonly used in many applied problems, particularly in lifetime data analysis [1]. The Gompertz distribution has been applied in the analysis of survival, in some sciences such as gerontology [2], computer [3], biology [4], and marketing science [5]. The hazard rate function of the Gompertz distribution is an increasing function and often applied to describe the distribution of adult life spans by actuaries and demographers [6].

New families of distributions are produced day by day and are useful for adding parameters to all forms of probability distributions which makes the resulting distribution more flexible for modeling heavily skewed dataset. Some of these families of distributions include the beta generalized family (Beta-G) by Eugene et al. [7], Transmuted family of distributions by Shaw and Buckley [8], Gamma-G (type 1) by Zografos and Balakrishnan [9], the Kumaraswamy-G by Cordeiro and de Castro [10], McDonald-G by Alexander et al. [11], Gamma-G (type 2) by Ristic et al. [12], Gamma-G (type 3) by Torabi and Montazari [13], Log-gamma-G by Amini et al. [14], Exponentiated T-X by Alzaghal et al. [15], Exponentiated-G (EG) by Cordeiro et al. [16], Weibull-X by Alzaatreh et al. [17], Weibull-G by Bourguignon et al. [18], Logistic-G by Torabi and Montazari [19], Gamma-X by Alzaatreh et al. [20], a Lomax-G family by Cordeiro et al. [21], a new generalized Weibull-G family by Cordeiro et al. [22], a Beta Marshall-Olkin family of distributions by Alizadeh et al. [23], Logistic-X by Tahir et al. [24], a new Weibull-G family by Tahir et al. [25], a Lindley-G family by Cakmakyapan and Ozel [26], a Gompertz-G family by Alizadeh et al. [27] and Odd Lindley-G family by Gomes-Silva et al. [28] and so on.

Following the introduction of the above listed families of probability distribution and the desire to add skewness and flexibility to classical distributions particularly the Gompertz distribution, many authors have proposed different extensions of the distribution and some of the recent and known studies include the generalized Gompertz distribution by El-Gohary and Al-Otaibi [29] which was based on an idea of Gupt and Kundu [30], the Beta Gompertz distribution by Jafaril et al. [31], the odd generalized Exponential-Gompertz distribution by El-Damcese et al. [32], the Transmuted Gompertz distribution by Abdul-Moniem and Seham [33] and the Lomax-Gompertz distribution by Omale et al. [34].

It has been discovered that using power transformation of a random variable offers a more flexible distribution model by adding a new parameter called the power parameter. Ghitany et al. [35] introduced two parameters distribution called power Lindley distribution and this model provides more flexibility than Lindley distribution. Also Rady et al. [36] introduced a three parameter Power Lomax Distribution using the power transformation approach. The new distribution exhibited a much more flexible model for life time data especially bladder cancer data than its predecessor Lomax distribution with a decreasing, inverted bath tub hazard rate function. They also used a real life data to illustrate and compare the potential of power Lomax distribution with other competing distributions and the results showed that it offered a better fit than a set of extensions of Lomax distribution.

Hence, our interest in this article is to present another extension of the Gompertz distribution using the power transformation approach considered previously by Ghitany et al. [35] and Rady et al., [36] and hope that it will yield a better model for analyzing real life situations especially in survival analysis.

The cumulative distribution function (cdf) of the Gompertz distribution with parameters α *and* β and the probability density function (pdf) is given as:

$$G(x)=1-e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)} \quad (1)$$

and

$$g(x)=\alpha e^{\beta x} e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)} \quad (2)$$

respectively. For $x \geq 0, \alpha > 0, \beta > 0$ where α *and* β are scale and shape parameters of the model respectively.

The remaining parts of this article are presented in sections as follows: definition of the new distribution with its graphical analysis is provided in section 2. Section 3 derived some properties of the new distribution such as limiting behavior, quantile function for median, skewness and kurtosis as well as simulation of random variables, survival and hazard functions and distribution of order statistics. The estimation of parameters using maximum likelihood estimation (MLE) is provided in section 4. An application of the new model with other existing distributions to a dataset on the remission times of a random sample of 128 bladder cancer patients is done in section 5 and a useful summary and conclusion is given in section 6.

2 Formulation of the Power Gompertz Distribution (PGD)

2.1 Definition

Here we introduce a new extension of the Gompertz distribution by considering the power transformation, $X = T^{\frac{1}{\theta}}$, where the random variable T is said to follow a Gompertz distribution with parameters α *and* β. The distribution of X is referred to as Power Gompertz distribution. Symbolically, it is abbreviated by $X \sim PGD(\alpha, \beta, \theta)$ to indicate that the random variable X has the power Gompertz distribution with parameters α, β *and* θ.

Therefore, the cumulative distribution function (cdf) of the Power Gompertz distribution (PGD) with parameters α, β *and* θ and the probability density function (pdf) of the Power Gompertz distribution (PGD) are given as:

$$F(x)=1-e^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)} \quad (3)$$

and

$$f(x)=\alpha\theta x^{\theta-1} e^{\beta x^{\theta}} e^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)} \quad (4)$$

respectively. For $x > 0, \alpha > 0, \beta > 0, \theta > 0$ where α *and* β are scale and shape parameters of the model respectively and θ is the power parameter responsible for addition of skewness and flexibility into the conventional Gompertz distribution.

2.2 Graphical Presentation of Pdf and Cdf of PGD

The *pdf* and *cdf* of the PGD using some parameter values are displayed in Figs. 1 and 2 respectively as follows.

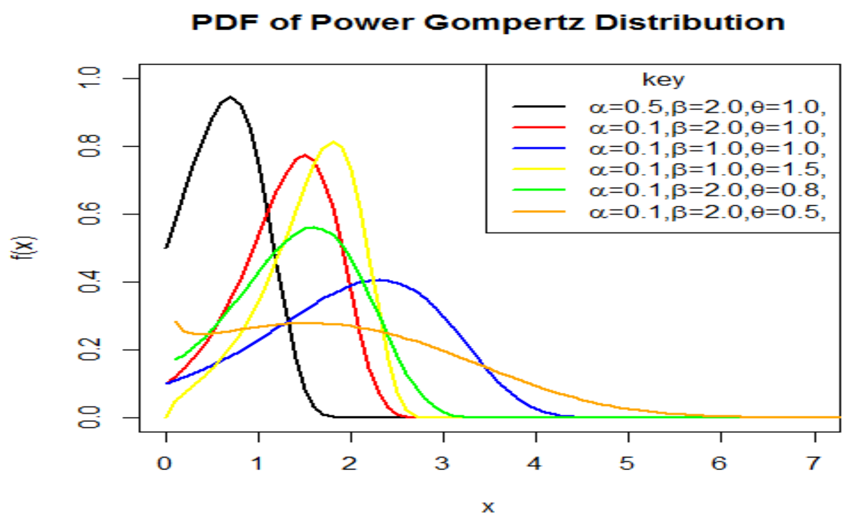

Fig. 1. *PDF* of the PGD for different values of the parameters

Fig. 1 indicates that the PGD distribution is positively skewed and takes various shapes depending on the parameter values.

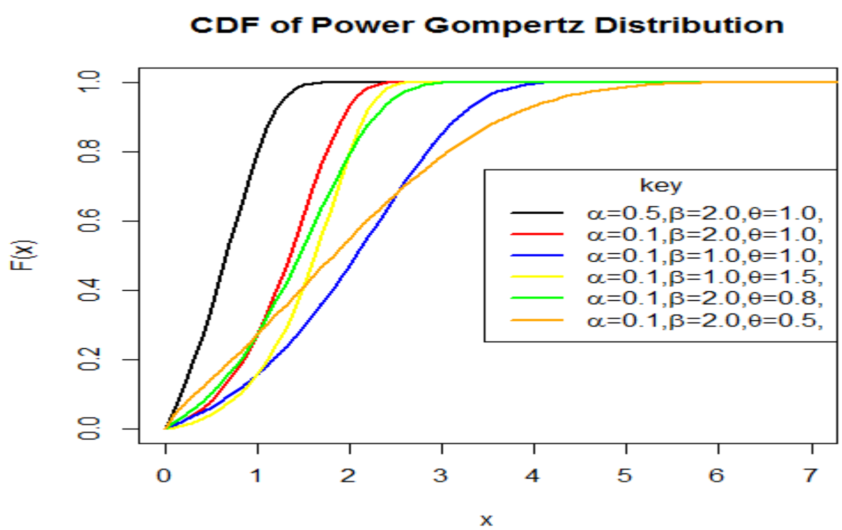

Fig. 2. *CDF* of the PGD for different values of the parameters

Also, from the above *cdf* plot in Fig. 2, it is clear that the *cdf* approaches one (1) when X tends to infinity and equals zero when X tends to zero as normally expected.

3 Mathematical and Statistical Properties of PGD

In this section, we derived, study and discuss some properties of the PGD distribution. They are as follows:

3.1 Asymptotic behavior

This section investigates the limiting behavior of the PGD, that is, the limit of the *PDF* and *CDF* of the PGD as X approaches infinity, $x \to \infty$ and as X tends to zero, $x \to 0$. This is demonstrated as follows:

For the PDF:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \alpha\theta x^{\theta-1} e^{\beta x^{\theta}} e^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)} = \alpha\theta(\infty)^{\theta-1} e^{\beta(\infty)^{\theta}} e^{-\frac{\alpha}{\beta}\left(e^{\beta(\infty)^{\theta}}-1\right)} = 0 \quad (5)$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \alpha\theta x^{\theta-1} e^{\beta x^{\theta}} e^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)} = \alpha\theta(0)^{\theta-1} e^{\beta(0)^{\theta}} e^{-\frac{\alpha}{\beta}\left(e^{\beta(0)^{\theta}}-1\right)} = 0 \quad (6)$$

For the CDF:

$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty} \left\{1 - e^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)}\right\} = 1 - e^{-\frac{\alpha}{\beta}\left(e^{\beta(\infty)^{\theta}}-1\right)} = 1 - 0 = 1 \quad (7)$$

$$\lim_{x\to 0} F(x) = \lim_{x\to 0} \left\{1 - e^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)}\right\} = 1 - e^{-\frac{\alpha}{\beta}\left(e^{\beta(0)^{\theta}}-1\right)} = 1 - 1 = 0 \quad (8)$$

This demonstration above affirms that the distribution has at least one mode or it is a unimodal distribution and that it is a valid probability distribution.

3.2 Quantile function

Hyndman and Fan [37] defined the quantile function for any distribution in the form $Q(u) = X_q = F^{-1}(u)$ where $Q(u)$ is the quantile function of $F(x)$ for $0 < u < 1$

Taking *F(x)* to be the *cdf* of the PGD and inverting it as above will give us the quantile function as follows:

$$F(x) = 1 - e^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)} = u \quad (9)$$

Simplifying equation (9) above and solving for X presents the quantile function of the PGD as:

$$Q(u) = X_q = \sqrt[\theta]{\frac{1}{\beta}\log\left(1 - \frac{\beta}{\alpha}(1-u)\right)} \quad (10)$$

This function is derived above is used for obtaining some moments like skewness and kurtosis as well as the median and for generation of random variables from the distribution in question.

3.3 Skewness and Kurtosis

This paper presents the quantile based measures of skewness and kurtosis due to non-existence of the classical measures in some cases.

The Bowley's measure of skewness (Kennedy and Keeping [38]) based on quartiles is given by;

$$SK = \frac{Q\left(\frac{3}{4}\right) - 2Q\left(\frac{1}{2}\right) + Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)} \tag{11}$$

And the Moor's kurtosis by Moors [39] is on octiles and is given by;

$$KT = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) - Q\left(\frac{3}{8}\right) + \left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{1}{8}\right)} \tag{12}$$

Where $Q(.)$ is obtainable with the help of equation (10).

3.4 Reliability analysis of the PGD

The Survival function describes the likelihood that a system or an individual will not fail after a given time. Mathematically, the survival function is given by:

$$S(x) = 1 - F(x) \tag{13}$$

Applying the *cdf* of the PGD in (13), the survival function for the PGD is obtained as:

$$S(x) = 1 - \left\{ 1 - e^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}} - 1\right)} \right\}$$

$$S(x) = e^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}} - 1\right)} \tag{14}$$

The following is a plot for the survival function of the PGD using different parameter values as shown in Fig. 3 below;

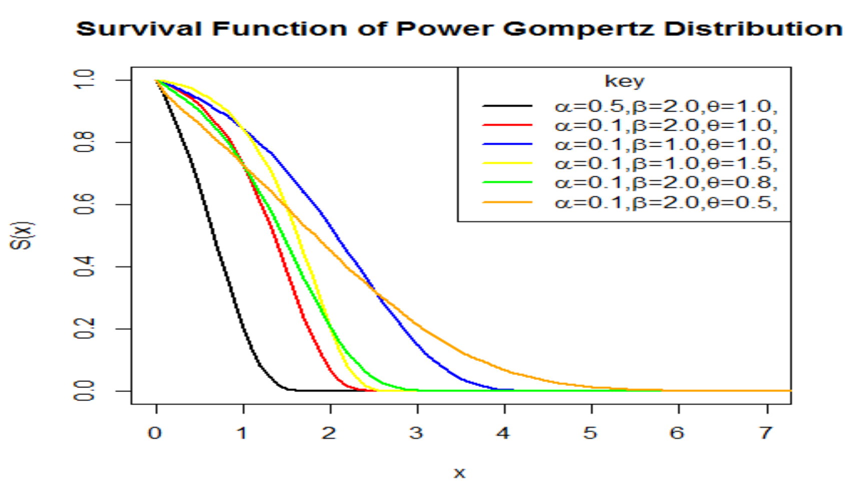

Fig. 3. Survival function of the PGD at different parameter values

The plots in Fig. 3 shows that the probability of survival equals one (1) at initial time or early age and it decreases as time increases and equals zero (0) as time approaches infinity.

Hazard function is the probability that a component will fail or die for an interval of time. The hazard function is defined as;

$$h(x) = \frac{f(x)}{S(x)} = \frac{f(x)}{1 - F(x)} \tag{15}$$

Meanwhile, the expression for the hazard rate of the PGD is given by

$$h(x) = \frac{\alpha\theta x^{\theta-1} e^{\beta x^{\theta}} e^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)}}{1 - \left\{1 - e^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)}\right\}}$$

$$h(x) = \alpha\theta x^{\theta-1} e^{\beta x^{\theta}} \tag{16}$$

where $x, \alpha, \beta, \theta > 0$.

The following is a plot of the hazard function for arbitrary parameter values in Fig. 4.

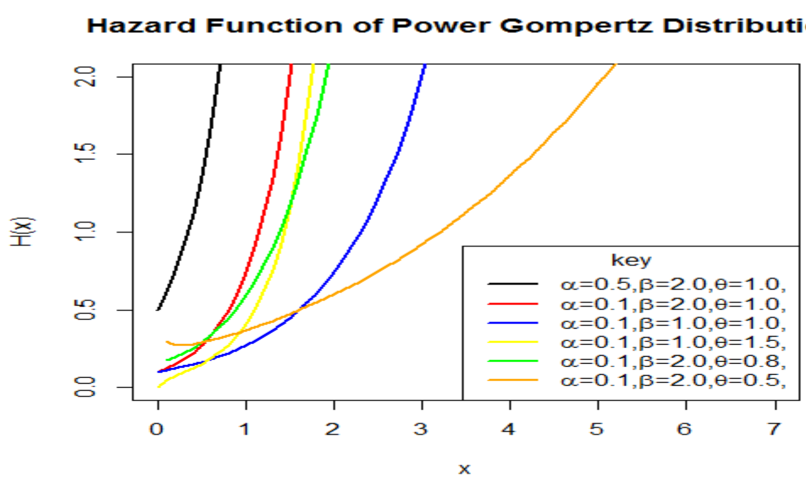

Fig. 4. The hazard function of the PGD for different values of the parameters as displayed in the key on the plots

The figure above revealed that the PGD has increasing failure rate which implies that the probability of failure for any random variable following a PGD increases as time increases, that is, probability of failure or death increases as life ages.

3.5 Order statistics

Suppose $X_1, X_2, ..., X_n$ is a random sample from the PGD and let $X_{1:n}, X_{2:n}, ..., X_{i:n}$ denote the corresponding order statistic obtained from this same sample. The *pdf*, $f_{i:n}(x)$ of the i^{th} order statistic can be obtained by

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}\sum_{k=0}^{n-i}(-1)^k\binom{n-i}{k}f(x)F(x)^{k+i-1} \quad (17)$$

Using (3) and (4), the *pdf* of the i^{th} order statistics $X_{i:n}$, can be expressed from (17) as;

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}\sum_{k=0}^{n-i}(-1)^k\binom{n-i}{k}\left[\alpha\theta x^{\theta-1}e^{\beta x^{\theta}}e^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)}\right]\left[1-e^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)}\right]^{i+k-1} \quad (18)$$

Hence, the *pdf* of the minimum order statistic $X_{(1)}$ and maximum order statistic $X_{(n)}$ of the PGD are respectively given by;

$$f_{1:n}(x)=n\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\left[\alpha\theta x^{\theta-1}e^{\beta x^{\theta}}e^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)}\right]\left[1-e^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)}\right]^{k} \quad (19)$$

and

$$f_{n:n}(x)=n\left[\alpha\theta x^{\theta-1}e^{\beta x^{\theta}}e^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)}\right]\left[1-e^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)}\right]^{n-1} \quad (20)$$

4 Estimation of Unknown Parameters of the PGD Using Method of Maximum Likelihood

In this section, the estimation of the parameters of the PGD is done by using the method of maximum likelihood estimation (MLE). Let $X_1, X_2, \ldots\ldots\ldots, X_n$ be a sample of size 'n' independently and identically distributed random variables from the PGD with unknown parameters α, β *and* θ defined previously.
The likelihood function of the PGD using the pdf in equation (4) is given by;

$$L\left(\underline{X}/\alpha,\beta,\theta\right)=\left(\alpha\theta\right)^n\prod_{i=1}^{n}\left(x_i^{\theta-1}\right)e^{\beta\sum_{i=1}^{n}x_i^{\theta}}e^{-\frac{\alpha}{\beta}\sum_{i=1}^{n}\left(e^{\beta x_i^{\theta}}-1\right)} \quad (21)$$

Let the natural logarithm of the likelihood function be, $l(\eta)=\log L\left(\underline{X}\mid\alpha,\beta,\theta\right)$, therefore, taking the natural logarithm of the function above gives:

$$l(\eta)=n\log\alpha+n\log\theta+(\theta-1)\sum_{i=1}^{n}\log x+\beta\sum_{i=1}^{n}x_i^{\theta}-\frac{\alpha}{\beta}\sum_{i=1}^{n}\left(e^{\beta x_i^{\theta}}-1\right) \quad (22)$$

Differentiating $l(\eta)$ partially with respect to α, β *and* θ respectively gives the following results;

$$\frac{\partial l(\eta)}{\partial\alpha}=\frac{n}{\alpha}-\frac{1}{\beta}\sum_{i=1}^{n}\left(e^{\beta x_i^{\theta}}-1\right) \quad (23)$$

$$\frac{\partial l(\eta)}{\partial \beta} = \sum_{i=1}^{n} x_i^{\theta} - \frac{\alpha}{\beta} \sum_{i=1}^{n} \left\{ x_i^{\theta} e^{\beta x_i^{\theta}} - \frac{\alpha}{\beta^2} \left(e^{\beta x_i^{\theta}} - 1 \right) \right\} \tag{24}$$

$$\frac{\partial l(\eta)}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \log x + \beta \sum_{i=1}^{n} x_i^{\theta} \ln x_i - \alpha \sum_{i=1}^{n} x_i^{\theta} \ln x_i e^{\beta x_i^{\theta}} \tag{25}$$

Making equation (23), (24) and (25) equal to zero (0) and solving for the solution of the non-linear system of equations produce the maximum likelihood estimates of parameters $\hat{\alpha}, \hat{\beta}$ and $\hat{\theta}$. However, these solutions cannot be obtained manually except numerically with the aid of suitable statistical software like *R*, *SAS*, *MATHEMATICA e.t.c.* Hence, some datasets are being considered in the next section to fit the proposed distribution with other distributions using "Adequacy Model" package in R software.

5 Applications to Three Real Life Datasets

This section presents a real life dataset, its descriptive statistics, graphical summary and applications. The section compares the fits of the Power Gompertz Distribution (PGD) and Gompertz Distribution (GD) using a dataset on the remission times of a random sample of 128 bladder cancer patients.

To compare the above listed distributions, we have considered some model selection criteria which include the value of the log-likelihood function evaluated at the MLEs (ℓ), Akaike Information Criterion, *AIC,* Consistent Akaike Information Criterion, *CAIC*, Bayesian Information Criterion, *BIC* and Hannan Quin Information Criterion, *HQIC*. These statistics are computed with the following formulas:

$$AIC = -2\ell + 2k, \quad BIC = -2\ell + k\log(n), \quad CAIC = -2\ell + \frac{2kn}{(n-k-1)} \text{ and } HQIC = -2\ell + 2k\log\left[\log(n)\right]$$

Where ℓ denotes the value of log-likelihood function evaluated at the *MLEs*, *k* is the number of model parameters and *n* is the sample size. Meanwhile, when taking our decisions we consider any model with the lowest values for these statistics to be a best model that fit the dataset. The required computations are carried out using the R package "AdequacyModel" which is freely available from http://cran.r-project.org/web/packages/AdequacyModel/AdequacyModel.pdf.

Table 2 list the Maximum Likelihood Estimates of the model parameters whereas the statistics AIC, CAIC, BIC and HQIC for the fitted PGD and GD models are given in Tables 3 based on the dataset on the remission times of a random sample of 128 bladder cancer patients.

Dataset: This data represents the remission times (in months) of a random sample of 128 bladder cancer patients adopted from the work of Rady et al. [36]. It has previously been used by Lee and Wang [40], Rady et al. [36], Ieren and Chukwu [41] and Abdullahi et al. [42]. It is given and summarized as follows:

0.080, 0.200, 0.400, 0.500, 0.510, 0.810, 0.900, 1.050, 1.190, 1.260, 1.350, 1.400, 1.460, 1.760, 2.020, 2.020, 2.070, 2.090, 2.230, 2.260, 2.460, 2.540, 2.620, 2.640, 2.690, 2.690, 2.750, 2.830, 2.870, 3.020, 3.250, 3.310, 3.360, 3.360, 3.480, 3.520, 3.570, 3.640, 3.700, 3.820, 3.880, 4.180, 4.230, 4.260, 4.330, 4.340, 4.400, 4.500, 4.510, 4.870, 4.980, 5.060, 5.090, 5.170, 5.320, 5.320, 5.340, 5.410, 5.410, 5.490, 5.620, 5.710, 5.850, 6.250, 6.540, 6.760, 6.930, 6.940, 6.970, 7.090, 7.260, 7.280, 7.320, 7.390, 7.590, 7.620, 7.630, 7.660, 7.870, 7.930, 8.260, 8.370, 8.530, 8.650, 8.660, 9.020, 9.220, 9.470, 9.740, 10.06, 10.34, 10.66, 10.75, 11.25, 11.64, 11.79, 11.98, 12.02, 12.03, 12.07, 12.63, 13.11, 13.29, 13.80, 14.24, 14.76, 14.77, 14.83, 15.96, 16.62, 17.12, 17.14, 17.36, 18.10, 19.13, 20.28, 21.73, 22.69, 23.63, 25.74, 25.82, 26.31, 32.15, 34.26, 36.66, 43.01, 46.12, 79.05.

Table 1. Summary statistics for the data set

Parameters	n	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
Values	128	0.0800	3.348	6.395	11.840	9.366	79.05	110.425	3.3257	19.1537

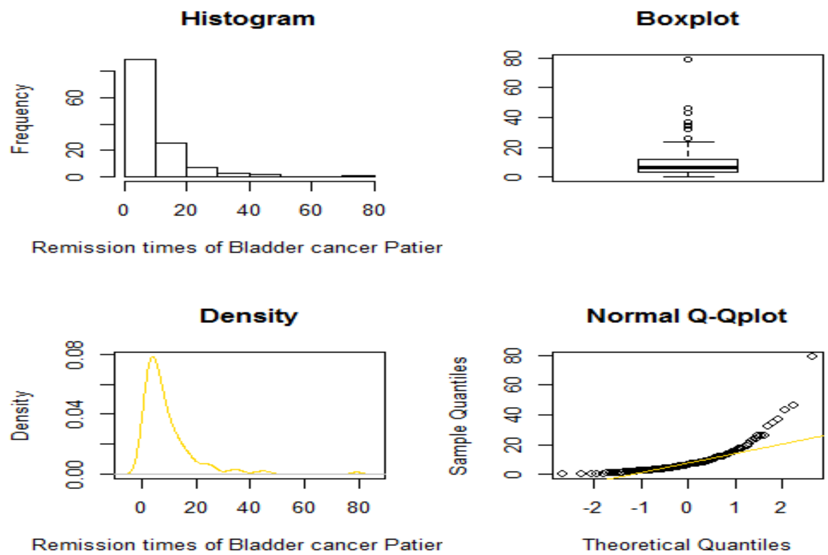

Fig. 5. A graphical summary of the aforementioned dataset

Based on the descriptive statistics in Table 1 and the histogram, box plot, density and normal Q-Q plot generally known as graphical summary shown in Fig. 5 above, it is seen that the dataset on the remission times of a random sample of 128 bladder cancer patients is heavily skewed to the right or positively skewed.

Table 2. Maximum likelihood parameter estimates for the dataset

Distribution	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$
PGD	0.071803	-0.009980	1.200525
GD	0.114844	-0.007515	-

Table 3. The statistics ℓ, AIC, CAIC, BIC and HQIC for the dataset

Distribution	$\hat{\ell}$	***AIC***	***CAIC***	***BIC***	***HQIC***	**Ranks**
PGD	-411.1731	828.3463	836.9023	828.5397	831.8226	1^{st}
GD	-413.7624	831.5248	837.2289	831.6208	833.8424	2^{nd}

The following figure displayed the histogram and estimated densities and cdfs of the fitted models to a dataset on the remission times of a random sample of 128 bladder cancer patients.

Table 3 presents the parameter estimates and the values of AIC, CAIC, BIC and HQIC for the PGD and GD using a dataset on the remission times of a random sample of 128 bladder cancer patients which is skewed to the right. The values of AIC, CAIC, BIC and HQIC in Table 3 are smaller for the PGD compared to those of the GD and these results indicate that the Power Gompertz distribution (PGD) is better than the Gompertz distribution (GD) and this is confirmed from the estimated density plots in Fig. 6 as well as the Q-Q plots presented in Fig. 7. This result confirms the that fact that the power transformation approach of adding parameter to distributions has advantage over the conventional probability distributions. These results above has proven that the power parameter is really responsible for additional skewness and flexibility in some continuous probability distributions just previously reported by Ghitany et al. [35] and Rady et al. [36].

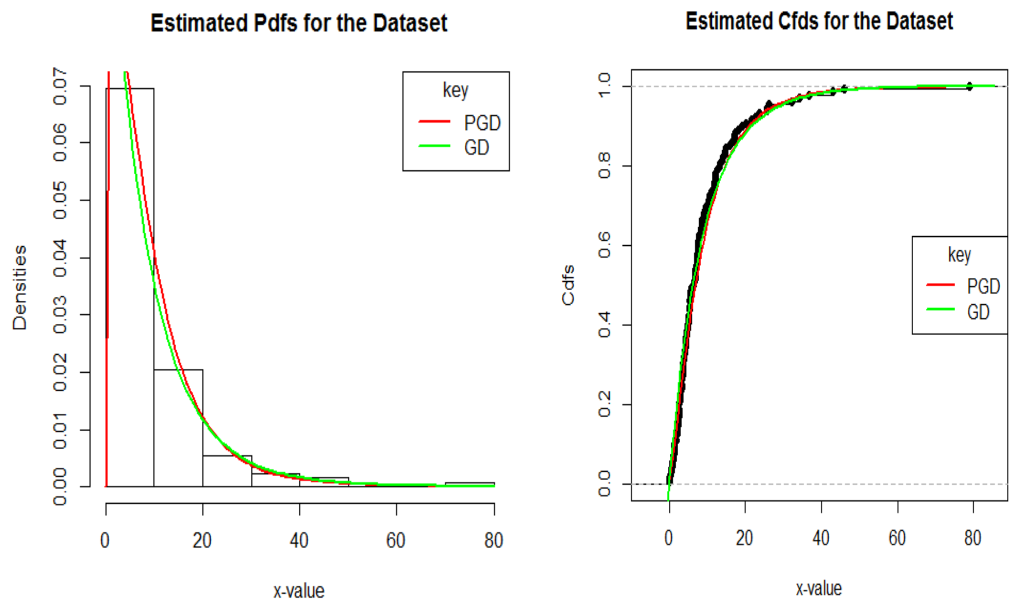

Fig. 6. Histogram and plots of the estimated densities and cdfs of the fitted distributions to a dataset on the remission times of a random sample of 128 bladder cancer patients

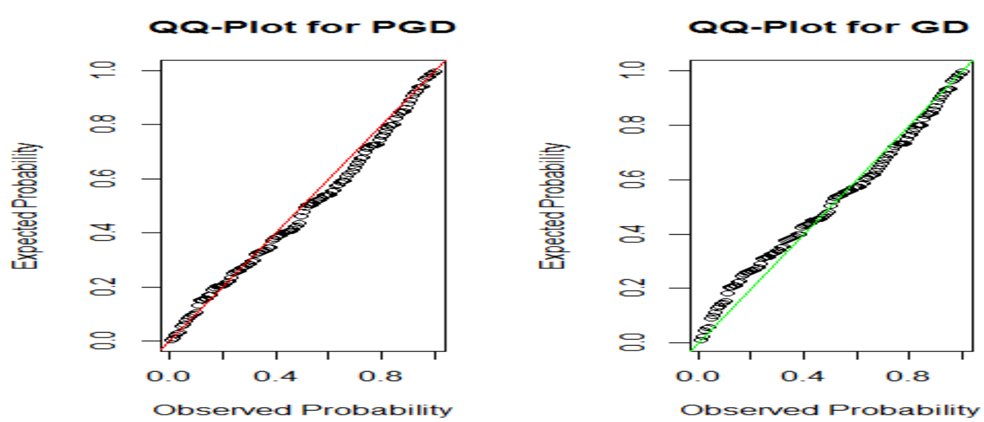

Fig. 7. Probability plots for the fit of the PGD and GD based on our dataset on the remission times of a random sample of 128 bladder cancer patients

6 Summary and Conclusion

This research considered a power transformation approach to define and study a Gompertz distribution leading to a new distribution called "Power Gompertz distribution". The research derived and studied some of its properties of the proposed distribution with graphical analysis and discussion on its usefulness and applications. The paper has checked analytically the validity of the Power Gompertz distribution with plots of its probability density function and cumulative distribution function which has shown that the new distribution is flexible with different shapes and could be adequately used in real life situations. We also checked the limiting behavior of the new model which has also proven that it is a very accurate probability model. This study also derived and generated plots of the survival and hazard function which has revealed that the new distribution will be useful for modeling random variables whose chances of survival decreases with time and those of failure increases with time, that our model has a decreasing hazard rate useful for many lifetime datasets. This article also proposed density functions for minimum and maximum order statistics which are applicable in robust statistical estimation and detection of outliers, characterization of probability distributions and goodness of fit tests, entropy estimation, analyses of censored samples, reliability analysis, quality control and strength of materials. Hence, having demonstrated earlier in the previous section, it is clear that the new model *(PGD)* has a better fit compared to the Gompertz distribution based on the data set considered in this study.

Competing Interests

Authors have declared that no competing interests exist.

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