



Ultimate Boundedness of Discrete-Time Uncertain Neural Networks with Leakage and Time-Varying Delays

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

In this paper, by using the general discrete Halanay inequalities, the techniques of inequalities and some other properties, we study the ultimate boundedness of a class of the discrete-time uncertain neural network systems and obtain several sufficient conditions to ensure the ultimate boundedness of discrete-time uncertain neural networks with leakage and time-varying delays. Finally numerical examples are given to verify the correctness of the conclusion.

Keywords: Discrete-time neural network; ultimate boundedness; time-varying delay; uncertainty.

1 Introduction

Neural network is an artificial system that simulates the function of human brain nervous system, and develops an artificial system with a variety of intelligent information processing functions by exploring the neurons in the human brain [1-3]. Since the discovery of neural network in the 1940s, many domestic and foreign scholars have devoted themselves to it, which has developed a broad prospect for the research and development of neural network. At present, on the basis of the study of network model and algorithm, artificial neural network has been

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used to establish a number of practical application systems, such as making robots, completing some signal processing or pattern recognition functions.

However, it can not be ignored that neural network, as a nonlinear system, will have time delay in the process of hardware implementation. Therefore, in the neural network system, the existence of time delay may reduce the transmission speed of the neural network, and affect the dynamic behavior of the whole neural network. In [4], S. Li studied non-autonomous recursive neural networks with variable time delay and obtained the boundedness and global exponential stability. In [5], Arslan discussed the robust stability for a class of neural networks with multiple time delays. In [6], Y. Cheng considered the stability analysis of the DNNs with time-varying delay. And in [7-9], scholars have also explored other characteristics of time delay neural networks. In addition, leakage delay is also inevitable and more difficult to deal with. Therefore, considering leakage delay is of great help to the study of neural networks. In [10-13], scholars have studied the leakage delay.

Furthermore, in the process of establishing the neural network model, due to modeling errors, external disturbances and parameter fluctuations, the value of network parameters may show deviation. These uncertainties in the network parameters may lead to some complex dynamics phenomena in the neural network. The existence of uncertainty can also make the system difficult to control, and even difficult to model. For this reason, scholar has analyzed the robust stability of time-varying delay neural networks with norm bounded uncertainties in [14]. Similarly, scholars also conducted research and analysis on the system of uncertainty from different perspectives in [15-18].

With the deepening of the research on neural networks, the categories of neural networks have been refined. Discrete neural network is a kind of artificial neural network, which has the functions of bidirectional associative memory, nonlinear output adjustment and adaptive tracking. However, in the current dynamic behavior of neural networks, most dynamic behaviors involve continuous time systems, while discrete-time dynamic neural networks are more relevant to many problems in nature and biological reality than continuous time networks. Hence, in practical application, discrete-time neural network is actually more important than continuous time neural network. Thereupon, it is particularly necessary and important to study the dynamic characteristics of discrete-time neural networks. As an important dynamic characteristic in the study of neural networks, stability analysis of neural networks has been studied by many scholars [19-24]. While there are few researches on the ultimate boundedness of discrete-time neural networks.

Difference equations are often used to help model real life problems as a special case of inequality [25]. This article will use the differential equation to establish the model of uncertain discrete-time neural network. And by using the general discrete Halanay inequalities, the techniques of inequalities and some other properties, we obtain some sufficient conditions to ensure the ultimate boundedness of uncertain discrete-time neural networks, and numerical examples are given to prove the validity of the theoretical results.

2 Preliminary Knowledge

For the purposes of proof and derivation, we need to clarify some symbols, definitions, and preliminary results:

Let R denote the set of all real numbers, R^+ the set of positive real numbers, R_0^+ the set of non-negative real numbers, Z the set of integers, Z^+ the set of positive integers, Z^{-r} the set of all integers greater than or equal to $-r$. And $\|A\|$ the norm in space $R^{n \times n}$, $\|A\|_1$ the column norm in space $R^{n \times n}$, A^T the transpose of the matrix, $diag(a_i)$ the diagonal matrix.

Also $y(\cdot) = [y_1(\cdot), \dots, y_p(\cdot)]^T \in R^p$ is the neuron state vector, $b = [b_1, \dots, b_p]^T$ is a constant input vector. $\varphi_i(k), i = 1, 2, \dots, r$ are positive integers and represent the time-varying delays satisfying $0 \leq \varphi_i(k) \leq \varphi_i$, $i = 1, 2, \dots, r$, and $\varphi_i \geq 0$ are known integers. $\omega_i(k), i = 1, 2, \dots, r$ are the leakage delays satisfying $0 \leq \omega_i(k) \leq \omega_i$, $i = 1, 2, \dots, r$, and $\omega_i \geq 0$ are known integers. $A_i = diag(a_{mi})(a_{mi} \in (0, 1))$,

$A = \text{diag}(A_i)(A_i \in (0,1))$, B , B_i and C_i are the interconnection matrices. $g(\cdot) = [g_1(y_1(\cdot)), g_2(y_2(\cdot)), \dots, g_p(y_p(\cdot))]^T$ is the activation function, and we assume that there exists a constant $L_m \geq 0$ such that the activation function $g_m(\cdot), m = 1, 2, \dots, p$ satisfies the following inequality $|g_m(x) - g_m(y)| \leq L_m |x - y|, \forall x, y \in R$.

At the same time, in order to satisfy the uncertainty conditions of the system, we establish some uncertainty matrixs with the following conditions $\Delta A_i(k), \Delta A(k), \Delta B(k)$ and $\Delta C_i(k), i = 1, 2, \dots, r$:

$$\Delta A_i(k) = H_{1_i} F(k) E_{1_i}, \Delta A(k) = H_1 F(k) E_1,$$

$$\Delta B(k) = H_2 F(k) E_2, \Delta C_i(k) = W_i F(k) D_i,$$

where $H_{1_i}, H_1, H_2, E_{1_i}, E_1, E_2, D_i$ and $W_i, i = 1, 2, \dots, r$ are known matrices, and $F(k)$ is an unknown time-varying matrix satisfying

$$\|F(k)\| \leq I,$$

where I is the identity matrix. Besides, $\max_m L_m$ is the maximum of the vector and

$$l = \max_m (L_m), a_{\max} = \max_m (A_m),$$

$$e_{1\max} = \max_m (E_{1m}), e_{2\max} = \max_m (E_{2m}), d_{i\max} = \max_m (D_{im}).$$

We mainly study the ultimate boundedness of uncertain discrete-time neural network system, so it is necessary to give the definition of the ultimate boundedness of uncertain discrete-time neural network system:

Definition 2.1(Exponential stability) [19]. The system is called exponentially stable if for any solution $y(k, \phi)$ with the initial condition ϕ , there exist constant scalars $0 < \varepsilon < 1$ and $\beta > 0$ such that

$$\|y(k)\| \leq \beta \|\phi\| \varepsilon^k, \forall k \geq 0,$$

where $\|\phi\| = \max_{s \in [-\rho_r, 0]} \{\|\phi(s)\|\}$ for all admissible uncertainties.

Definition 2.2(Ultimate boundedness). The system is called ultimate bounded if for any solution $y(k, \phi)$ with the initial condition ϕ , there exist constant scalars $0 < \varepsilon < 1$ and $\beta > 0$ such that

$$\|y(k)\| \leq \beta \|\phi\| \varepsilon^k + M, \forall k \geq 0,$$

where M is a constant and $\|\phi\| = \max_{s \in [-\rho_r, 0]} \{\|\phi(s)\|\}$ for all admissible uncertainties.

3 Ultimate Boundedness

In order to obtain the sufficient conditions for the ultimate boundedness of uncertain discrete-time neural networks quickly and effectively, the difference equation is used to establish the model, and the discrete Halanay

inequality and the skills of inequality are used to obtain the sufficient conditions for the ultimate boundedness. Now, we consider the following DNNs with leakage and time-varying delays:

$$y(k+1) = \sum_{i=1}^r [A_i + \Delta A_i(k)] y(k - \omega_i(k)) + [B + \Delta B(k)] g(y(k)) + \sum_{i=1}^r [C_i + \Delta C_i(k)] g(y(k - \varphi_i(k))) + b. \tag{3-1}$$

with the initial conditions for the system

$$y_m(s) = \phi_m(s), \quad m = 1, 2, \dots, p,$$

where $\phi(s) = [\phi_1(s), \phi_2(s), \dots, \phi_p(s)]^T$ is an initial function, $s \in [-\rho_r, 0]$, and ρ_r is the maximum value of φ_r and ω_r .

To more simple and clear application of the system, $y(k)$ is expressed by y_k . We will use the inequalities in Lemma 3.1 to prove Theorem 3.2, thus obtaining sufficient conditions for the ultimate boundedness of uncertain discrete-time neural networks with leaky and time-varying delays. And when b is the zero vector, the condition of ultimate boundedness can be generalized to the condition of exponential stability.

Lemma 3.1. Let $\sum_{i=0}^r q_i(\alpha_i + 1) < p \leq 1$, and let $\{v_j\}_{j \in \mathbb{Z}^{-\alpha_r}}$ be a sequence of real numbers, where $q_i \in \mathbb{R}_0^+, \alpha_i \in \mathbb{Z}^+, i = 1, \dots, r; p, \alpha_r \in \mathbb{R}^+, 0 = \alpha_0 < \alpha_1 < \dots < \alpha_r$. Assume the following inequality holds

$$\square v_k \leq -pv_k + \sum_{i=0}^r q_i(\alpha_i + 1) \sum_{j=0}^{\alpha_i} v_{k-j} + \xi, \quad k \in \mathbb{Z}^0,$$

where $\Delta v_k = v_{k+1} - v_k$, and ξ is a constant. Then there exists $\lambda_0 \in (0, 1)$ such that

$$v_k \leq \max \{0, v_0, v_{-1}, \dots, v_{-\alpha_r}\} \lambda_0^k + \frac{\xi}{p - \sum_{i=0}^r q_i(\alpha_i + 1)}, \quad k \in \mathbb{Z}^0.$$

Besides, λ_0 may be chosen as the smallest root of the polynomial

$$P(\lambda) = \lambda^{\alpha_r+1} - (1-p)\lambda^{\alpha_r} - \sum_{i=0}^r q_i(\alpha_i + 1) \sum_{j=0}^{\alpha_i} \lambda^{\alpha_r-j}$$

which lies in $(0, 1)$.

Proof. See [26].

Now, Lemma 3.1 will be used to prove and obtain sufficient conditions for the ultimate boundedness of uncertain discrete-time neural networks with leakage and time-varying delays.

Theorem 3.2. The system (3-1) is ultimate bounded if

$$l\|B\| + l\|H_2\|e_{2\max} + \sum_{i=1}^r (\rho_i + 1)(l(\|C_i\| + \|W_i\|d_{i\max}) + a_{i\max} + \|H_{1_i}\|e_{1_i\max}) < 1. \tag{3-2}$$

Proof. Consider the function $z_k = \|y_k\|$. Then the difference equation in system (3-1) can be expressed as

$$\begin{aligned} \Delta z(k) &= \|y_{k+1}\| - \|y_k\| \\ &= \left\| \sum_{i=1}^r (A_i + \Delta A_i)y_{k-\omega_i(k)} + (B + \Delta B_k)g(y_k) + \sum_{i=1}^r (C_i + \Delta C_{k_i})g(y_{k-\varphi_i(k)}) + b \right\| - \|y_k\| \\ &\leq \sum_{i=1}^r \left\| (A_i + \Delta A_i)y_{k-\omega_i(k)} \right\| + \left\| (B + \Delta B_k)g(y_k) \right\| \\ &\quad + \sum_{i=1}^r \left\| (C_i + \Delta C_{k_i})g(y_{k-\varphi_i(k)}) \right\| + \|b\| - \|y_k\| \\ &\leq \sum_{i=1}^r (\|A_i\| + \|H_{1_i}F_kE_{1_i}\|) \|g_{k-\omega_i(k)}\| + l(\|B\| + \|H_2F_kE_2\|) \|y_k\| \\ &\quad + l \sum_{i=1}^r (\|C_i\| + \|W_iF_kD_i\|) \|y_{k-\varphi_i(k)}\| + \|b\| - \|y_k\| \\ &\leq \sum_{i=1}^r (\|A_i\| + \|H_{1_i}\|e_{1_i\max}) \|y_{k-\omega_i(k)}\| + l(\|B\| + \|H_2\|e_{2\max}) \|y_k\| \\ &\quad + l \sum_{i=1}^r (\|C_i\| + \|W_i\|d_{i\max}) \|y_{k-\varphi_i(k)}\| + \|b\| - \|y_k\| \\ &\leq -(1 - l\|B\| - l\|H_2\|e_{2\max}) \|y_k\| + \sum_{i=1}^r (a_{i\max} + \|H_{1_i}\|e_{1_i\max}) \|y_{k-\omega_i(k)}\| \\ &\quad + l \sum_{i=1}^r (\|C_i\| + \|W_i\|d_{i\max}) \|y_{k-\varphi_i(k)}\| + \|b\| \\ &\leq -(1 - l\|B\| - l\|H_2\|e_{2\max}) \|y_k\| \\ &\quad + \sum_{i=1}^r \sum_{j=0}^{\rho_i} \left[l(\|C_i\| + \|W_i\|d_{i\max}) + a_{i\max} + \|H_{1_i}\|e_{1_i\max} \right] \|y_{k-j}\| + \|b\| \\ &\leq -(1 - l\|B\| - l\|H_2\|e_{2\max}) \|y_k\| \\ &\quad + \sum_{i=1}^r (\rho_i + 1) \left[l(\|C_i\| + \|W_i\|d_{i\max}) + a_{i\max} + \|H_{1_i}\|e_{1_i\max} \right] \sum_{j=0}^{\rho_i} \|y_{k-j}\| + \|b\| \\ &= -pz_k + \sum_{i=0}^r (\rho_i + 1)q_i \sum_{j=0}^{\rho_i} z_{k-j} + \xi, \end{aligned}$$

where $p = 1 - l\|B\| - l\|H_2\|e_{2\max}$, $q_i = l(\|C_i\| + \|W_i\|d_{i\max}) + a_{i\max} + \|H_{1_i}\|e_{1_i\max}$, $\xi = \|b\|$.

According to Lemma 3.1, there must be $\lambda_0 \in (0, 1)$ such that

$$z_k \leq \max \{0, z_0, z_{-1}, \dots, z_{-\rho_r}\} \lambda_0^k + \frac{\xi}{p - \sum_{i=0}^r q_i(\alpha_i + 1)}, k \in Z^0.$$

Therefore, we provide

$$\begin{aligned} z_k &= \|y_k\| \leq \max \{0, z_0, z_{-1}, \dots, z_{-\rho_r}\} \lambda_0^k + \frac{\xi}{p - \sum_{i=0}^r q_i(\alpha_i + 1)}, k \in Z^0. \\ &= \max \{0, \|y_0\|, \|y_{-1}\|, \dots, \|y_{-\alpha_r}\|\} \lambda_0^k + \frac{\xi}{p - \sum_{i=0}^r q_i(\alpha_i + 1)} \\ &\leq \|\phi\| \lambda_0^k + \frac{\xi}{p - \sum_{i=0}^r q_i(\alpha_i + 1)} = \|\phi\| \lambda_0^k + j, \end{aligned}$$

where $j = \frac{\xi}{p - \sum_{i=0}^r q_i(\alpha_i + 1)}$ is a constant.

From definition 2.2, system (3-1) can be proved ultimate bounded.

Lemma 3.3 [19]. Let $\sum_{i=0}^r q_i(\alpha_i + 1) < p \leq 1$, and let $\{v_j\}_{j \in Z^{-\alpha_r}}$ be a sequence of real numbers, where $q_i \in R_0^+, \alpha_i \in Z^+, i = 1, \dots, r; p, \alpha_r \in R^+, 0 = \alpha_0 < \alpha_1 < \dots < \alpha_r$. Assume the following inequality holds

$$\square v_k \leq -pv_k + \sum_{i=0}^r q_i(\alpha_i + 1) \sum_{j=0}^{\alpha_i} v_{k-j}, k \in Z^0,$$

where $\Delta v_k = v_{k+1} - v_k$. Then there exists $\lambda_0 \in (0, 1)$ such that

$$v_k \leq \max \{0, v_0, v_{-1}, \dots, v_{-\alpha_r}\} \lambda_0^k, k \in Z^0.$$

Besides, λ_0 may be chosen as the smallest root of the polynomial

$$P(\lambda) = \lambda^{\alpha_r+1} - (1-p)\lambda^{\alpha_r} - \sum_{i=0}^r q_i \sum_{j=0}^{\alpha_i} \lambda^{\alpha_r-j}$$

which lies in $(0, 1)$.

Proof. See [26].

Now, Lemma 3.3 will be used to prove and obtain that when b is the zero vector, the exponential stability condition of the system can be deduced.

Corollary 3.4 [19]. Assuming b in system (3-1) is the zero vector, then the system (3-1) is exponentially stable if

$$l\|B\| + l\|H_2\|e_{2\max} + \sum_{i=1}^r (\rho_i + 1)(l(\|C_i\| + \|W_i\|d_{i\max}) + a_{i\max} + \|H_{1_i}\|e_{1_i\max}) < 1, \tag{3-3}$$

Proof. See [19].

Since time-varying delay is more common than leakage delay, the conclusion will be more general if we only study time-varying delay. Therefore, the system will degraded to an uncertain discrete-time neural network with no leakage delay.

So the system (3-1) is going to be equal to

$$y(k+1) = [A + \Delta A(k)]y(k) + [B + \Delta B(k)]g(y(k)) + \sum_{i=1}^r [C_i + \Delta C_i(k)]g(y(k - \varphi_i(k))) + b, \tag{3-4}$$

with the initial conditions for the system

$$y_m(s) = \phi_m(s), \quad m = 1, 2, \dots, p,$$

where $\phi(s) = [\phi_1(s), \phi_2(s), \dots, \phi_p(s)]^T$ is an initial function, $s \in [-\varphi_r, 0]$, and φ_r is the maximum value of φ_i .

To prove and obtain sufficient conditions for the ultimate boundedness of uncertain discrete-time neural networks with time-varying delays, lemma 3.5 needs to be introduced to prepare for its implementation.

Lemma 3.5. Let $\sum_{i=0}^r q_i(\beta_i + 1) < p \leq 1$ and let $\{v_j\}_{j \in Z^{-\beta_r}}$ be a sequence of real numbers, where $q_i \in R_0^+, \beta_i \in Z^+, i = 1, \dots, r; p, \beta_r \in R^+, 0 = \beta_0 < \beta_1 < \dots < \beta_r$. Assume the following inequality holds

$$\square v_k \leq -pv_k + \sum_{i=0}^r q_i(\beta_i + 1) \sum_{j=0}^{\beta_i} v_{k-j} + \xi, \quad k \in Z^0,$$

where $\Delta v_k = v_{k+1} - v_k$, and ξ is a constant. Then there exists $\lambda_0 \in (0, 1)$ such that

$$v_k \leq \max \{0, v_0, v_{-1}, \dots, v_{-\beta_r}\} \lambda_0^k + \frac{\xi}{p - \sum_{i=0}^r q_i(\beta_i + 1)}, \quad k \in Z^0.$$

Besides, λ_0 may be chosen as the smallest root of the polynomial

$$P(\lambda) = \lambda^{\beta_r+1} - (1-p)\lambda^{\beta_r} - \sum_{i=0}^r q_i(\beta_i+1) \sum_{j=0}^{\beta_i} \lambda^{\beta_r-j}$$

which lies in (0,1).

Proof. See [26].

Now, Lemma 3.5 is used to prove and obtain the sufficient conditions for the ultimate boundedness of uncertain discrete-time neural networks with time-varying delays.

Theorem 3.6. The system (3-4) is ultimate bounded if

$$a_{\max} + \|H_1\|e_{1\max} + l\|B\| + l\|H_2\|e_{2\max} + l\sum_{i=1}^r (\varphi_i + 1)(\|C_i\| + \|W_i\|d_{i\max}) < 1, \tag{3-5}$$

Proof. Consider the function $z_k = \|y_k\|$. Then the difference equation in system (3-4) can be expressed as

$$\begin{aligned} \Delta z(k) &= \|y_{k+1}\| - \|y_k\| \\ &= \left\| (A + \Delta A_k)y_k + (B + \Delta B_k)g(y_k) + \sum_{i=1}^r (C_i + \Delta C_{k_i})g(y_{k-\varphi_i(k)}) + b \right\| - \|y_k\| \\ &\leq \|(A + \Delta A_k)y_k\| + \|(B + \Delta B_k)g(y_k)\| + \sum_{i=1}^r \|(C_i + \Delta C_{k_i})g(y_{k-\varphi_i(k)})\| + \|b\| - \|y_k\| \\ &\leq (a_{\max} + \|H_1 F_k E_1\|)\|y_k\| + l(\|B\| + \|H_2 F_k E_2\|)\|y_k\| \\ &\quad + l\sum_{i=1}^r (\|C_i\| + \|W_i F_k D_i\|)\|y_{k-\varphi_i(k)}\| + \|b\| - \|y_k\| \\ &\leq (a_{\max} + \|H_1\|e_{1\max})\|y_k\| + l(\|B\| + \|H_2\|e_{2\max})\|y_k\| \\ &\quad + l\sum_{i=1}^r (\|C_i\| + \|W_i\|d_{i\max})\|y_{k-\varphi_i(k)}\| + \|b\| - \|y_k\| \\ &\leq -(1 - a_{\max} - \|H_1\|e_{1\max} - l\|B\| - l\|H_2\|e_{2\max})\|y_k\| \\ &\quad + l\sum_{i=1}^r (\|C_i\| + \|W_i\|d_{i\max})\|y_{k-\varphi_i(k)}\| + \|b\| \\ &\leq -(1 - a_{\max} - \|H_1\|e_{1\max} - l\|B\| - l\|H_2\|e_{2\max})\|y_k\| \\ &\quad + l\sum_{i=1}^r \sum_{j=0}^{\kappa_i} (\|C_i\| + \|W_i\|d_{i\max})\|y_{k-j}\| + \|b\| \\ &\leq -(1 - a_{\max} - \|H_1\|e_{1\max} - l\|B\| - l\|H_2\|e_{2\max})\|y_k\| \\ &\quad + l\sum_{i=1}^r (\kappa_i + 1)(\|C_i\| + \|W_i\|d_{i\max})\sum_{j=0}^{\kappa_i} \|y_{k-j}\| + \|b\| \\ &= -pz_k + \sum_{i=0}^r (\kappa_i + 1)q_i \sum_{j=0}^{\kappa_i} z_{k-j} + \xi. \end{aligned}$$

where $p = 1 - a_{\max} - \|H_1\|e_{1\max} - l\|B\| - l\|H_2\|e_{2\max}$, $q_i = \|C_i\| + \|W_i\|d_{i\max}$, $\xi = \|b\|$.

According to Lemma 3.5, there must be $\lambda_0 \in (0, 1)$ such that

$$z_k \leq \max \{0, z_0, z_{-1}, \dots, z_{-\kappa_r}\} \lambda_0^k + \frac{\xi}{p - \sum_{i=0}^r q_i(\beta_i + 1)}, k \in Z^0.$$

Therefore, we provide

$$\begin{aligned} z_k = \|y_k\| &\leq \max \{0, z_0, z_{-1}, \dots, z_{-\kappa_r}\} \lambda_0^k + \frac{\xi}{p - \sum_{i=0}^r q_i(\beta_i + 1)}, k \in Z^0. \\ &= \max \{0, \|y_0\|, \|y_{-1}\|, \dots, \|y_{-\kappa_r}\|\} \lambda_0^k + \frac{\xi}{p - \sum_{i=0}^r q_i(\beta_i + 1)} \\ &\leq \|\phi\| \lambda_0^k + \frac{\xi}{p - \sum_{i=0}^r q_i(\beta_i + 1)} = \|\phi\| \lambda_0^k + j, \end{aligned}$$

where $j = \frac{\xi}{p - \sum_{i=0}^r q_i(\beta_i + 1)}$ is a constant.

From definition 2.2, system (3-4) can be proved ultimate bounded.

4 Numerical Example

Example 3.1. Consider the DNNs (3-1) with $r = 1$ where

$$\begin{aligned} A_0 = A_1 &= \begin{pmatrix} 0.12 & 0 \\ 0 & 0.11 \end{pmatrix}, B = \begin{pmatrix} 0.05 & -0.08 \\ -0.08 & 0.03 \end{pmatrix}, C_0 = C_1 = \begin{pmatrix} 0.08 & -0.05 \\ -0.05 & 0.07 \end{pmatrix}, \\ H_{1_0} = H_{1_1} &= \begin{pmatrix} 0.012 & -0.003 \\ 0.003 & -0.013 \end{pmatrix}, H_2 = \begin{pmatrix} 0.03 & -0.005 \\ 0.02 & 0.011 \end{pmatrix}, \\ W_0 = W_1 &= \begin{pmatrix} -0.011 & 0.015 \\ -0.012 & 0.016 \end{pmatrix}, E_{1_0} = E_{1_1} = (-0.016 \quad 0.012), \\ E_2 &= (-0.012 \quad 0.01), D_0 = (0.013 \quad -0.02), D_1 = (0.01 \quad -0.013), \\ F(k) &= \begin{pmatrix} \sin(k) \\ -\sin(k) \end{pmatrix}, b = (-3 \quad 5), \varphi_0(k) = 1 - \sin(\frac{k\pi}{2}), \varphi_1(k) = 2 - \sin(\frac{k\pi}{2}), \\ \omega_0(k) &= 1 - \cos(\frac{k\pi}{2}), \omega_1(k) = 2 - \cos(\frac{k\pi}{2}) \\ g_1(s) &= \cos(-0.4s) + 0.2 \sin(s), g_2(s) = \tanh(0.2s). \end{aligned}$$

Then we can get $\|B\|_1=0.13, \|C_0\|_1=\|C_1\|_1=0.13, \|H_{1_0}\|_1=\|H_{1_1}\|_1=0.016, \|H_2\|_1=0.05,$

$$\|W_0\|_1 = \|W_1\|_1 = 0.031, \varphi_0 = 0, \omega_0 = 1, \rho_0 = \max\{\varphi_0, \omega_0\} = 1, \varphi_1 = 1, \omega_1 = 2, \rho_1 = \max$$

$$\{\varphi_1, \omega_1\} = 2, a_{0\max} = a_{1\max} = 0.12, e_{1_0\max} = e_{1_1\max} = 0.016, e_{2\max} = 0.012, d_{0\max} = 0.02,$$

$d_{1\max} = 0.013, L_1 = 0.4, L_2 = 0.5$ and $l = 0.5$. By calculation, we have

$$0.5\|B\|_1 + 0.5\|H_2\|_1 e_{2\max} + (\rho_0 + 1)(0.5(\|C_0\|_1 + \|W_0\|_1 d_{0\max}) + a_{0\max} + \|H_{1_0}\|_1 e_{1_0\max}) \\ + (\rho_1 + 1)(0.5(\|C_1\|_1 + \|W_1\|_1 d_{1\max}) + a_{1\max} + \|H_{1_1}\|_1 e_{1_1\max}) = 0.9928045 < 1,$$

which satisfied (3-2) in Theorem 3.2. Thus, the system (3-1) is ultimate bounded.

Example 3.2. Consider the DNNs (3-1) with $r = 1$ where

$$A_0 = A_1 = \begin{pmatrix} 0.11 & 0 \\ 0 & 0.13 \end{pmatrix}, B = \begin{pmatrix} 0.05 & -0.08 \\ -0.08 & 0.02 \end{pmatrix}, C_0 = C_1 = \begin{pmatrix} 0.06 & -0.05 \\ -0.05 & 0.02 \end{pmatrix},$$

$$H_{1_0} = H_{1_1} = \begin{pmatrix} 0.006 & -0.007 \\ 0.007 & -0.011 \end{pmatrix}, H_2 = \begin{pmatrix} 0.01 & -0.005 \\ 0.012 & 0.016 \end{pmatrix},$$

$$W_0 = W_1 = \begin{pmatrix} -0.009 & 0.013 \\ -0.011 & 0.017 \end{pmatrix}, E_{1_0} = E_{1_1} = (-0.013 \quad 0.015),$$

$$E_2 = (-0.013 \quad 0.011), D_0 = (0.05 \quad -0.03), D_1 = (0.017 \quad -0.012),$$

$$F(k) = \begin{pmatrix} \sin(k) \\ -\sin(k) \end{pmatrix}, b = (0 \quad 0), \varphi_0(k) = 1 - \sin\left(\frac{k\pi}{2}\right), \varphi_1(k) = 2 - \sin\left(\frac{k\pi}{2}\right),$$

$$\omega_0(k) = 1 - \cos\left(\frac{k\pi}{2}\right), \omega_1(k) = 2 - \cos\left(\frac{k\pi}{2}\right)$$

$$g_1(s) = \cos(-0.4s) + 0.2 \sin(s), g_2(s) = \tanh(0.2s).$$

Then we can get $\|B\|_1=0.13, \|C_0\|_1=\|C_1\|_1=0.11, \|H_{1_0}\|_1=\|H_{1_1}\|_1=0.018,$

$$\|H_2\|_1=0.022, \|W_0\|_1 = \|W_1\|_1 = 0.03, \varphi_0 = 0, \omega_0 = 1, \rho_0 = \max\{\varphi_0, \omega_0\} = 1, \varphi_1 = 1, \omega_1 = 2,$$

$$\rho_1 = \max\{\varphi_1, \omega_1\} = 2, a_{0\max} = a_{1\max} = 0.13, e_{1_0\max} = e_{1_1\max} = 0.015, e_{2\max} = 0.011,$$

$d_{0\max} = 0.05, d_{1\max} = 0.012, L_1 = 0.4, L_2 = 0.5$ and $l = 0.5$. By calculation, we have

$$0.5\|B\|_1 + 0.5\|H_2\|_1 e_{2\max} + (\rho_0 + 1)(0.5(\|C_0\|_1 + \|W_0\|_1 d_{0\max}) + a_{0\max} + \|H_{1_0}\|_1 e_{1_0\max}) \\ + (\rho_1 + 1)(0.5(\|C_1\|_1 + \|W_1\|_1 d_{1\max}) + a_{1\max} + \|H_{1_1}\|_1 e_{1_1\max}) = 0.993511 < 1,$$

which satisfied (3-3) in Corollary 3.4. Thus, the system (3-1) is exponentially stable.

Example 3.3. Consider the DNNs (3-4) with $r = 1$ where

$$\begin{aligned}
 A &= \begin{pmatrix} 0.18 & 0 \\ 0 & 0.15 \end{pmatrix}, B = \begin{pmatrix} 0.17 & -0.07 \\ -0.07 & 0.15 \end{pmatrix}, C_0 = \begin{pmatrix} 0.18 & -0.05 \\ -0.05 & 0.18 \end{pmatrix}, \\
 C_1 &= \begin{pmatrix} 0.2 & -0.2 \\ -0.1 & 0.1 \end{pmatrix}, H_1 = \begin{pmatrix} 0.01 & -0.023 \\ 0.003 & -0.013 \end{pmatrix}, H_2 = \begin{pmatrix} 0.003 & -0.005 \\ 0.002 & 0.011 \end{pmatrix}, \\
 W_0 &= \begin{pmatrix} -0.011 & 0.013 \\ -0.012 & 0.006 \end{pmatrix}, W_1 = \begin{pmatrix} -0.011 & 0.015 \\ -0.019 & 0.006 \end{pmatrix}, E_1 = (-0.013 \quad 0.017), \\
 E_2 &= (-0.012 \quad 0.021), D_0 = (0.011 \quad -0.023), D_1 = (0.014 \quad -0.016), \\
 F(k) &= \begin{pmatrix} \sin(k) \\ -\sin(k) \end{pmatrix}, b = (-2 \quad 1), \varphi_0(k) = 1 + \sin\left(\frac{k\pi}{2}\right), \varphi_1(k) = 2 + \sin\left(\frac{k\pi}{2}\right), \\
 g_1(s) &= \cos(-0.4s) + 0.2\sin(s), g_2(s) = \tanh(0.2s).
 \end{aligned}$$

Then we can get $\|B\|_1 = 0.24, \|C_0\|_1 = 0.23, \|C_1\|_1 = 0.3, \|H_1\|_1 = 0.036, \|H_2\|_1 = 0.016,$

$$\begin{aligned}
 \|W_0\|_1 &= 0.023, \|W_1\|_1 = 0.03, \varphi_0 = 1, \varphi_1 = 2, a_{\max} = 0.18, e_{1\max} = 0.017, e_{2\max} = 0.021, d_{0\max} = \\
 &0.023, d_{1\max} = 0.016, L_1 = 0.4, L_2 = 0.5 \text{ and } l = 0.5. \text{ By calculation, we have} \\
 &a_{\max} + \|H_1\|_1 e_{1\max} + 0.5\|B\|_1 + 0.5\|H_2\|_1 e_{2\max} + 0.5(\varphi_0 + 1)(\|C_0\|_1 + \|W_0\|_1 d_{0\max}) \\
 &+ 0.5(\varphi_1 + 1)(\|C_1\|_1 + \|W_1\|_1 d_{1\max}) = 0.982029 < 1,
 \end{aligned}$$

which satisfied (3-5) in Theorem 3.6. Thus, the system (3-4) is ultimate bounded.

5 Conclusion

In this paper, by using the general discrete Halanay inequalities, the techniques of inequalities and some other properties, we obtain several sufficient conditions to ensure the ultimate boundedness of a class of discrete-time uncertain neural networks. However, due to the random phenomenon is ubiquitous in real life, therefore we will devote to the research on the ultimate boundedness of neural networks with more randomness in the future work.

Competing Interests

Authors have declared that no competing interests exist.

References

- [1] Huang S, Yang XF. Research Progress of Discrete Time Neural Network [J]. Application Research of Computers. 2009;26(08):2810-2812.
- [2] Zhu DQ. Research Status and Prospect of Artificial Neural Network [J]. Journal of Jiangnan University. 2004;01:103-110.

- [3] Xu XL. The development and current situation of artificial neural network [J]. *Microelectronics*. 2017;47(02):239-242.
- [4] Li SR. Boundedness and Global Exponential Stability of Non-autonomous Recurrent Neural Networks with Time-varying Delay [J]. *Journal of Baoji University of Arts and Science (Natural Science Edition)*. 2013;33(02):15-20.
- [5] Arslan E. Novel criteria for global robust stability of dynamical neural networks with multiple time delays, *Neural Networks*. 2021;142:119–127.
- [6] Cheng YF. Stability analysis of discrete-time neural network with time-varying delay [D]. *University of Electronic Science and Technology*; 2011.
- [7] Wang XH, Jiang MH. The Boundedness and Global Exponential Stability of Non-autonomous Neural Networks with Time-varying Delay [J]. *Journal of China Three Gorges University (Natural Science Edition)*. 2008;04:94-98.
- [8] Dong T. Analysis of Dynamic Behavior of Delayed Neural Network [D]. *Chongqing University*; 2013.
- [9] Jiang HJ, Teng ZD. The Boundedness and Stability of Non-autonomous Delayed Cellular Neural Networks [J]. *Journal of Natural Science of Heilongjiang University*. 2006;04:437-440.
- [10] Yan H, Zhao ZJ, Song QK. Global synchronization of complex-valued neural networks with leakage delay [J]. *Applied Mathematics and Mechanics*. 2016;37(08):832-841.
- [11] Liu LB, Pan HP. State Estimation of Complex Neural Networks with Leakage Delay and Mixed Additive Time-varying Delays [J]. *Applied Mathematics and Mechanics*. 2019;40(11):1246-1258.
- [12] Song Q, Cao J. Passivity of uncertain neural networks with both leakage delay and time-varying delay[J]. *Nonlinear Dynamics*. 2012;67(2):1695-1707.
- [13] Huang C, Cao J. Impact of leakage delay on bifurcation in high-order fractional BAM neural networks[J]. *Neural Networks*. 2018; 98:223-235.
- [14] Yücel E. An analysis of global robust stability of delayed dynamical neural networks[J]. *Neurocomputing*. 2015;165:436-443.
- [15] Li L. Robust predictive control of uncertain discrete-time systems [D]. *University of Science and Technology Beijing*; 2017.
- [16] Shan Y, Zhong S, Cui J, et al. Improved criteria of delay-dependent stability for discrete-time neural networks with leakage delay [J]. *Neurocomputing*. 2017;266:409-419.
- [17] Sun FQ. Stability analysis of time-varying uncertain time-delay systems [J]. *Journal of Jilin University (Information Science Edition)*. 2012;30(05):456-461.
- [18] Liu XY. Robust performance analysis of time-varying parameter uncertain time-delay systems [J]. *Knowledge Library*. 2019;22:35.
- [19] Suntonsinsoungvon E, Udpin S. Exponential stability of discrete-time uncertain neural networks with multiple time-varying leakage delays [J]. *Mathematics and Computers in Simulation*. 2020;171:233-245.
- [20] Zhu CL, Wang YM, Yi N. Stability Analysis of a Class of Discrete Time Neural Network [J]. *Journal of Yangzhou University (Natural Science Edition)*. 2014;17(04):20-24.
- [21] Xia F, Li XM. Global Exponential Stability of Discrete Time Neural Networks [J]. *Journal of Inner Mongolia Normal University (Natural Science Chinese Edition)*. 2006;01:12-16.

- [22] Liu Y, Wang Z, Liu X. Global exponential stability of generalized recurrent neural networks with discrete and distributed delays [J]. *Neural Networks*. 2006;19(5):667-675.
- [23] Cao J, Wang J. Global asymptotic and robust stability of recurrent neural networks with time delays [J]. *IEEE Transactions on Circuits and Systems I: Regular Papers*. 2005;52(2):417-426.
- [24] Zhou B, Song Q. Boundedness and complete stability of complex-valued neural networks with time delay [J]. *IEEE Transactions on Neural Networks and Learning Systems*. 2013;24(8):1227-1238.
- [25] Yang QX, Talking about the Application of Difference Equation [J], *Journal of Central University for Nationalities (Natural Science Edition)*. 2006;03:282-285.
- [26] Xu L. Generalized discrete Halanay inequalities and the asymptotic behavior of nonlinear discrete systems [J]. *Bulletin of the Korean Mathematical Society*. 2013;50(5):1555-1565.

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