Journal of Scientific Research & Reports

15(4): 1-6, 2017; Article no.JSRR.35451

ISSN: 2320-0227

n-co-coherent Modules

Abdul-Qawe Kaed[1*]

[1] *Department of Mathematics, Faculty of applied Science, University of Thamar, Republic of Yemen.*

Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/JSRR/2017/35451
Editor(s):
(1) Chun-Gang Zhu, School of Mathematical Sciences, Dalian University of Technology, China.
Reviewers:
(1) Piyush Shroff, Texas State University, USA.
(2) Francisco Bulnes, Technological Institute of High Studies of Chalco, Mexico.
Complete Peer review History: http://www.sciencedomain.org/review-history/20631

Received: 13th July 2017
Accepted 25th July 2017
Published 23rd August 2017

Original Research Article

ABSTRACT

In [1] the notion of n-coherent modules are introduced and studied. In this paper, we introduce and study a dual notion of n-coherent modules which we call it n-co-coherent modules.

Keywords: Finitely copresented modules; finitely generated modules; finitely presented modules; a finitely cogenerated module; a coherent module; a co-coherent module; a coherent ring and a co-coherent ring; n-coherent modules, n-co-coherent modules; n-coherent rings and n-co-coherent rings.

2010 Mathematics Subject Classification: 13E05, 13D02, 13E15, 13E99, 16D50, 16D80.

1 INTRODUCTION

Throughout this paper R means a commutative ring with an identity element and all modules are unital R-modules.

In [1] the notion of n-coherent modules is introduced and studied, such that for a ring R and a positive integer n, an R-module M is called n-coherent if M is n-present and each $(n-1)$-presented submodule of M is n-presented.

*Corresponding author: E-mail: Kaed2012d@gmail.com;

In this paper, we introduce and study a dual notion of n-coherent R-modules which we call it n-co-coherent R-modules, such that we define it as the following : For a ring R and a positive integer n, an R-module M is called n-co-coherent if M is n-copresent and each $(n-1)$-copresented submodule of M is n-copresented. If M is n-co-coherent modules for every positive integer n, we say that M is infinitely co-coheret. Recall that a module K is said to be n- presented, for some positive integer n, if there is an exact sequence of R-modules of the form

$$F_n \longrightarrow F_1 \longrightarrow .... \longrightarrow F_0 \longrightarrow K \longrightarrow 0$$

where F_i for $i = 0, 1, 2, ..., n$ are free and finitely generated modules. see [1] . A dual notion of n-presented is called n- copresented R-modules is defined as the following : For a ring R and a positive integer n, an R- module M is called n-copresented if there is an exact sequence of R-modules of the form

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow .... \longrightarrow I_n$$

where I_i for $i = 0, 1, 2, ...n$ are injective and finitely cogenerated modules.see [2] and [3]. Recall that an R-module M is called a finitely generated, if for any family $(M_i)_{i\in I}$ of submodules of M with $\Sigma_{i\in I} M_i = 0$, there is a finite subset J of I such that $\Sigma_{j\in J} M_j = 0$ (see [4, 5, 6]). A dual notion of a finitely generated is defined as the following : an R-module M is called a finitely cogenerated, if for any family $M_{i i\in I}$ of submodules of M with $\cap_{i\in I} M_i = 0$, there is a finite subset J of I such that $\cap_{j\in J} M_j = 0$, (see [4, 5, 6]).

As in the classical case, a finitely presented module M is defined as a module which is finitely generated such that, for every short exact sequence $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$, if L is a finitely generated, then K is also a finitely generated (see [5, 6]). Also a dual notion of a finitely presented is defined as the following: a finitely copresented module M is a finitely cogenerated such that, for every short exact sequence $0 \longrightarrow M \longrightarrow L \longrightarrow K \longrightarrow 0$, if L is a finitely cogenerated, then K is also a finitely cogenerated see [5, 6].

In proposition 2.1 we prove that if R is a ring and a positive integer n, then each $(n-1)$-copresented submodule of an n-co- coherent R-module is itself an n-co-coherent R-module. And we prove in proposition 2.2 that every n-co-coherent R-module is m-co- coherent for every positive integer $m \leq n$. In 2.3 we claim that for a positive integer n, if R is n-co-coherent ring, then every n-co-coherent R-module is an infinitely co-coherent, where For a positive integer n, a ring R is called n-co-coherent, if then every $(n-1)$-copresented ideal of R is n-copresented. That is equivallante that n-copresented R-module is $(n+1)$-copresented [2] section 3. The proposition 4.4 shows that 1-cocoherent module is just, the co-coherent module. In theorem 4.1 we give the main result which studies the behavior of n-co-coherent modules on short exact sequences. In corollary 3.2 we prove that every finite direct sum of an n-co-coherent R-modules are also an n-co-coherent R-modules. In theorem 3.3we prove that if $m \geq n$ are positive integers and let

$$M_0 \overset{u_1}{\to} M_1 \overset{u_2}{\longrightarrow} M_3 \overset{u_3}{\longrightarrow} .... \overset{u_m}{\to} M_m$$

be an exact sequence of an n-co-coherent R-modules. Then $Im(u_i)$, $Ker(u_i)$ and $Coker(u_i)$ are n-co-coherent R-modules for each $i = 1, 2, ...m$. In lemma we introduce an important result with the change of rings, such that If $H \longrightarrow G$ is a ring homomorphism such that G is $(n-1)$-copresented H-module, where $n \geq 1$ is a positive integer. And let M be G-module. If M is an n-co- coherent as H-module, then it is an n-co-coherent as G-module. Finally, in example3.5 we study an example to prove that if R is an n-co-coherent module, then $R|I$ is an n-co-coherent module.

2 n-CO-COHERENT R-MODULES

Definition 2.1. For a ring R and a positive integer $n \geq 1$, an R-module M is called n-co-coherent if M is n-copresent and each $(n-1)$-copresented submodule of M is n-copresented.

Remark 2.1. If M is n-co-coherent modules for every positive integer n, we say that M is infinitely co-coheret.

Proposition 2.1. *Let R be a ring and a positive integer n. Then each $(n-1)$-copresented submodule of n-co-coherent R-module is itself an n-co-coherent R-module*

Proof. Let M be an n-co-coherent R-module and let K be an $(n-1)$-copresented submodule of M, then by 2.1 K is an n-copresentnd. Suppose that L is $(n-1)$-copresented submodule of K, then it is is n-copresented because it is a submodule of M, hence K is an n-co-coherent R-module.

Proposition 2.2. *Every n-co-coherent R-module is m-co-coherent for every positive integer $m \leq n$.*

Proof. Suppose that M is n-co-coherent R-module, then M is n-copresented and for every $m \leq n$, M is m-copresented R-module. Suppose that K is $(n-1)$-copresented submodule of M, then K is $(m-1)$-copresented and we have K is n-copresented from 2.1 and for $m \leq n$,that implies that K is m-copresented and it follows that M is m-co-coherent.

Before we prove the following proposition we recall the following definition to use it. For a positive integer n, a ring R is called n-co-coherent, if then every $(n-1)$-copresented ideal of R is n-copresented. That is equivallante that n-copresented R-module is $(n+1)$-copresented. To know more about this concept see [2] section 3.

Proposition 2.3. *For a positive integer n, if R is n-co-coherent ring, then every n-co-coherent R-module is infinitely co-coherent.*

Proof. Therefore M is an n-co-coherent R-module, then it is an n-copresented and (since R is n-co-coherent ring), then from [2] proposition 3.2 then, M is an innitely copresented. Let N be an $(n-1)$-submodule of M, then it is an n-copresented (because M is an n- co-coherent R-module). And also that it is an infinitely copresented that is implies that for every positive integer $m \geq n, M$ is m- co-coherent R-module which means that M is an infinitely co-coherent module.

The following proposition shows that 1- co-coherent module is just, the co-coherent module.

Proposition 2.4. *For a ring R, an R-module is 1-co-coherent if and only if it is a co-coherent.*

Proof. : Suppose that M is 1-co-coherent R-module, Then M is 1-copresented this equivalente that M is a finitely copresented (see [2] proposition 2.3). Suppose that K is a finitey cogenerated submodule of M, from [2] proposition 2.2, we get K is 0-copresented R-module that is implies that K is 1-copresented see proposition 2.1 and it follows that K is a finitely copresented and from [7] 2.1, hence M is a co-coherent module.

Conversely. suppose that M is a co-coherent R-module, then it is a finitely copresented that is equivalents that M is 1-copresented see [2] proposition 2.3 . Suppose that K is 0-copresented submodule of M that is equivalentes that K is a finitely cogenerated submodule of M that is implies that from [7] 2.1 M is a finitely copresented and it follows that K is 1-copresented and from 2.1, hence M is 1-co-coherent R-module.

3 THE MAIN RESULTS

Now we give the main result in the following theorem which is a dual to a well known result on n-coherent modules see [1] and [3] and also study the behavior of n-co-coherent modules on an exact sequence.

Theorem 3.1. *Let R be a ring and let*

$$0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$$

be a short exact sequence of R-modules, Then for a positive integer n, we have:

1. *If A and C are n-co-coherent, then B is n-co-coherent.*
2. *If C is $(n-1)$-co-coherent and B is n-co-coherent, then A is n-co-coherent.*
3. *If A is $(n+1)$-co-coherent and B is n-co-coherent, then C is n-co-coherent.*
4. *If $B = A \oplus C$, then B is n-co-coherent if and only if A and C are n-co-coherent.*

1. Therefore A and C are n-co-coherent modules, then A and C are n-copresented modules. Thus, B is n-copresented module. Now let N be an $(n-1)$-copresented submodule of B, then we get the following daigram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & ker(\beta/L) & \xrightarrow{\alpha} & L & \xrightarrow{\beta} & \beta(L) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \end{array}$$

Let $Ker(\beta/N)$ and $\beta(L)$ be $(n-1)$-copresented submodules of A (resp of C), then they are n-copresented modules (Since A and C are n- co-coherent modules). tnen L is an n-copresented module (see 2.4(1) in [2]) and by 2.1 B is an n-co-coherent R-module as desired.

2. We have C is $(n-1)$-co-coherent and B is n-co-coherent, then C is $(n-1)$-copresented and B is n-copresented [2] proposition 2.4). Let K be an $(n-1)$-co-coherent submodule of A, then we have

$$0 \to K \xrightarrow{\alpha} \alpha(K) \xrightarrow{\beta} C \to 0$$

be a short exact sequence of R-modules and $\alpha(K)$ is n-co-coherent because it is a submodule of n-co-coherent R-module B so it is an n-copresented module and also C is an $(n-1)$-copresented module, hence K is an n-copresented module and it follows that A is an n-co-coherent module.

3. We have A is $(n+1)$-co-coherent and B are n-co-coherent modules, then from 2.1 A is $(n+1)$-copresented and B is n-copresented and hence C is n-copresented from [2] proposition 2.4). Let K be an $(n-1)$-copresent submodule of C and supose that $\beta^{-1}(K)$ is $(n-1)$-copresented submodule of B, then from 2.1 $\beta^{-1}(K)$ is n-co-coherent and so $\beta^{-1}(K)$ is n-copresent and we have

$$0 \to A \xrightarrow{\alpha} \beta^{-1}(K) \xrightarrow{\beta} K \to 0$$

be a short exact sequence of R-modules and from [2] proposition 2.4 K is an n-copresented module and it follows that C is an n-co-coherent module 2.1 .

4. Assume that A and C are n-co-coherents. Applying (1) to the following short exact sequence $0 \to A \to B = A \oplus C \to C \to 0$, we get that $A \oplus C$ is n-co-coherent module.

Conversely, suppose that $B = A \oplus C$ is n-co-coherent. Then from 2.1 B is n-copresent and also A and C are from [2] proposition 2.4). Let A_0 (resp C_0) be $(n-1)$-copresented submodules of A (resp C) , then A_0 and C_0 are n-copresented modules because they are submodules of $A \oplus C$ which is n-co-coherent, hence A and C are n-co-coherent modules.

Corollary 3.2. *Let R be a ring and $M_1, M_2......, M_n$ are R-modules, then $M_1, M_2......, M_n$ are n-co-coherents if and only if $\oplus_{i=1}^{n} M_i$ is an n-co-coherent R-module.*

Proof. Let $M_1, M_2......, M_n$ be n-co-coherent R-modules. We have a short exact sequence

$$0 \to M_n \to \oplus_{i=1}^{n} M_i \longrightarrow \oplus_{i=1}^{n-1} M_i \longrightarrow 0$$

and by induction if $n = 2$, then we get

$$0 \to M_2 \to \oplus_{i=1}^{2} M_i \longrightarrow M_1 \longrightarrow 0$$

from 4.1 (4) the asseration is true. Now we suppose that $M_1, M_2......, M_n$ are n-co-coherents if and only if $\oplus_{i=1}^{n} M_i$ is an n-co-coherent R-module and we prove it when n+1. The short exact

$$0 \to M_{n+1} \to \oplus_{i=1}^{n+1} M_i \longrightarrow M_1 \longrightarrow 0$$

and from 4.1 .2 that is implies that M_{n+1} is an n-co-coherent module (because M_1 is $(n-1)$-co-coherent). We have also

$$0 \to M_{n+1} \to \oplus_{i=1}^{n+1} M_i \longrightarrow \oplus_{i=1}^{n} M_i \longrightarrow 0,$$

then from 4.1 (4) $\oplus_{i=1}^{n+1} M_i$ is an n-co-coherent module and it follows that $M_1, M_2......, M_n$ are n-co-coherents if and only if $\oplus_{i=1}^{n} M_i$ is an n-co-coherent R-module for every n see [3].

Theorem 3.3. *Let $m \geq n$ be positive integers and let*

$$M_0 \overset{u_1}{\to} M_1 \overset{u_2}{\to} M_2 \overset{u_3}{\to} \cdots \overset{u_m}{\to} M_m$$

be an exact sequence of n-co-coherent R-modules. Then $Im(u_i)$, $Ker(u_i)$ and $Coker(u_i)$ are n-co-coherent R-modules for each $i = 1, 2, ..., m$.

Proof. : It suffices to prove the assertion for $m = n$. let

$$M_0 \xrightarrow{u_1} M_1 \xrightarrow{u_2} M_2 \xrightarrow{u_3} \cdots \xrightarrow{u_m} M_m$$

be an exact sequence of n-co-coherent R-modules. We have then exact sequences :

$$0 \longrightarrow Ker(u_1) \longrightarrow M_0 \longrightarrow Im(u_1) \longrightarrow 0$$

$$0 \rightarrow Im(u_i) = Ker(u_{i+1}) \rightarrow M_i \rightarrow Im(u_{i+1}) \rightarrow 0$$

for each $i = 1, 2, ..., n-1$, and

$$0 \longrightarrow Im(u_n) \longrightarrow M_n \longrightarrow Coker(u_n) \longrightarrow 0$$

and M_0 is finitely cogenerated (M_0 is n-co-coherent) and $Ker(u_1)$ is a submodule of M_0, then it is a finitely cogenerated; therefore, $Ker(u_2)$ is 1-copresented,and by induction, we conclude that $Ker(u_n)$ is $(n-1)$-copresented. And from 2.1 $Ker(u_n)$ is n-co-coherent module. Therefore $Ker(u_i)$ and $Im(u_i)$ are n-co-coherents by applying theorem 4.1 to the above exact seaunces. Finally, Theorem 4.1 and exactness of sequence

$$0 \longrightarrow Ker(u_i) \longrightarrow M_i \longrightarrow Coker(u_i) \longrightarrow 0$$

show that $Coker(u_i)$ are n-co-coherent modules.

We introduce a following important result with change of rings. see [3]

Lemma 3.4. *Let $H \longrightarrow G$ be a ring homomorphism such that G is $(n-1)$-copresented H-module, where $n \geq 1$ is a positive integer. Let M be G-module. If M is an n-co-coherent as H-module, then it is an n-co-coherent as G-module.*

Proof. Let $H \longrightarrow G$ be a ring homomorphism, such that G is $(n-1)$-copresented H-module. Let M be G-module such that M is an n-co-coherent as H-module. Then M is n-copresented as H-module [2] proposition 2.8 shows that M is an n-co-coherent as G- module. Let N be a submodule of M such that N is $(n-1)$-copresented as G-module. Then by [2] proposition 2.6 then N is $(n-1)$-copresented as H-module. Thus N is n- copresented as G-module since M is an n- co-coherent as H-module. Therefore N is n -copresented as G-module, hence by 2.1 M is an n-co-coherent as G-module.

Before we study the example we recall this definition: For a positive integer n, a ring R is called an n-co-coherent if and only if R is an n-co-coherent R-module.

Example 3.5. . *Let R be an n-co-coherent ring. And let I be an ideal of R such that I is $(n-1)$-copresented R-submodule of R-module. then I is $(n+1)$-copresented R-submodule (since R is an n-co-coherent ring). And we have*

$$0 \longrightarrow I \longrightarrow R \longrightarrow R|I \longrightarrow 0$$

and by 3.1 we have $R|I$ is an n-co-coherent module. In the case $n = 1$, then I is 0-copresented R-module that is implies that I is finitely cogenerted of a co-coherent ring R, then I from [7] by 2.3 is a co-coherent R-module and from [7] by 2.4 $R|I$ is a co- coherent R-module.

4 CONCLUSION

In this a studying is introduced the new concept in theory of modules (Linear algebra)

Definition 4.1. For a ring R and a positive integer $n \geq 1$, an R-module M is called n-co-coherent if M is n-copresent and each $(n-1)$-copresented submodule of M is n-copresented.

Remark 4.1. If M is n-co-coherent modules for every positive integer n, we say that M is infinitely co-coheret.

this concept is a dual notion of n-coherent modules and we called it n-co-coherent modules. It studies some properties of the concept.

Proposition 4.1. *Let R be a ring and a positive integer n. Then each $(n-1)$-copresented submodule of n-co-coherent R-module is itself an n-co-coherent R-module*

Proposition 4.2. *Every n-co-coherent R-module is m-co-coherent for every positive integer $m \leq n$.*

Proposition 4.3. *For a positive integer n, if R is n-co-coherent ring, then every n-co-coherent R-module is infinitely co-coherent.*

Proposition 4.4. *For a ring R, an R-module is 1-co-coherent if and only if it is a co-coherent.*

It gives the main result in the following theorem which is a dual to a well known result on n-coherent modules see [1] and also study the behavior of n-co-coherent modules on an exact sequence.

Theorem 4.1. *Let R be a ring and let*

$$0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$$

be a short exact sequence of R-modules, Then for a positive integer n, we have:

1. *If A and C are n-co-coherent, then B is n-co-coherent.*
2. *If C is $(n-1)$-co-coherent and B is n-co-coherent, then A is n-co-coherent.*
3. *If A is $(n+1)$-co-coherent and B is n-co-coherent, then C is n-co-coherent.*
4. *If $B = A \oplus C$, then B is n-co-coherent if and only if A and C are n-co-coherent.*

It explains that the sum of n-co-coherent is also n-co-coherent module :

Corollary 4.2. *Let R be a ring and $M_1, M_2......, M_n$ are R-modules, then $M_1, M_2......, M_n$ are n-co-coherents if and only if $\oplus_{i=1}^{n} M_i$ is an n-co-coherent R-module.*

Finally it studies and introduces very important result with change of rings.

Lemma 4.3. *Let $H \longrightarrow G$ be a ring homomorphism such that G is $(n-1)$-copresented H-module, where $n \geq 1$ is a positive integer. Let M be G-module. If M is an n-co-coherent as H-module, then it is an n-co-coherent as G-module.*

COMPETING INTERESTS

Author has declared that no competing interests exist.

References

[1] Dobbs DE, Kabbaj S, Mahdou N, Sobrani M. When is; $D + Mn$*-Coherent and an (n, d)-domain* . Lecture Notes in Pure and Appl. Math, Dekker. 1999;205:257-270.

[2] Bennis D, Bouzraa H, Kaed AQ. on n-Copresented modules and n-co-coherent rings. J. Electronic Journal Algebra(IEJA). 2012;12.

[3] Glaz S. Commutative coherent rings. Lecture Notes in Math, Springer-Verlag, Berlin; 1989.

[4] Andersen FW, Fuller KR. Rings and categories of modules. Spring- Verlag, Heidelberg, New York; 1974.

[5] Hiremath VA. Cofnitely generated and cofnitely related modules. Acta Math.Hung. 1982;39:1-9.

[6] Wisbauer R. Foundations of module and ring theory. Gordon and Breach, Reading; ©1991.

[7] Abdul-Qawe Kaed. A co-coherent module. United States of America Research Journal (USARJ) Copyright ©USARJ Publishing, Kansas, USA. 2014;3(1). ISSN 2332-2160

Peer-review history:
The peer review history for this paper can be accessed here:
http://sciencedomain.org/review-history/20631