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The Energy of Conjugate Graph of Some Finite Metabelian Groups of Order Less Than 24

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

The conjugacy classes of the Metabelian group G, plays an important role in defining the conjugate graph, whose vertices are non-central elements of G, and two vertices are connected if and only if they are conjugate. The constructions of conjugate graphs of all non abelian metabelian groups of order less than 24 are the basis for this paper. And the obtained results are then used to calculate the energy of the aforementioned group. This is aided by specialized programming software (maple).

Keywords: Metabelian groups; adjacency matrix; conjugacy class; conjugate graph; and energy of graph.

1 Introduction

According to Woods [1], the study of the energy of general simple graphs was first interpreted by Gutman in 1978, who was inspired by Huckel's Molecular Orbital Theory, which Huckel introduced in 1930. Chemists

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have used Huckel Molecular orbital theory to estimate the energies associated with $\pi-$electron orbitals in conjugated hydrocarbons.

Later in 1956, Gunthard and Primas realized that the Huckel's method is actually using the first-degree polynomial of matrix of a graph. Some few researchers have investigated the energy of graphs. Balakrishnan [2], for example, investigated the properties of graph energy, whereas Bapat and Pati [3] demonstrated that it is not possible for the energy of the graph to be an odd integer.

In that same year, Yu et al. (2006) developed different upper bounds for graph's energy. Pirzada and Gutman also displayed that the graph's energy cannot be the square root of an odd integer [4]. Bertram et al. [5] presented a graph related to group of conjugacy classes for conjugacy class graphs.

Rahman discover the metabelian groups with a maximum order of 24. [6].

There aren't as many researchers who use groups and expand their applications. Mohd [7] chose metabelian groups with a maximum order of 24 to determine its degree of commutativity. Halim [8] then broadened the implementation of metabelian groups with orders no greater than 24 to find the nth degree of commutativity.

The generalize conjugacy class graph of some finite non-abelian groups is initiated by Omer et al., [9]. Many researchers have attempted various approaches to working with metabelian groups of order less than 24, with some success, such as, Samin et al., [10], computed the number of conjugate classes of metabelian group of order less than 24. Ibrahim et al. (2019) present some graphs of finite metabelian groups of order less than 24, and also some examples of finite groups related with other graphs in their study. This paper considers ten metabelian groups of order less than 24, as determined by Rahman [6]. The groups are; D_3, D_4, Q = $Quarternion$, D_5, Z_3 x Z_4, A_4, D_5, D_6, D_7, D_8, Quasihedral – 16, Q_8, Z_2 x D_4, Q x Z_2, $Modular$ – 16, B, K, G_{44}, D_9, D_{10}, D_{11}, S_3 x Z_3, (Z_3 x Z_3) x Z_3, $Fr_{20} \cong Z_5$ x Z_4, Z_4 x Z_5, $Fr_{21} \cong Z_7$ x Z_3.

2 Preliminaries

Some definitions used in this work are as follows:

Definition 1.1

Let G be a finite group and H be its normal subgroup. The factor group of G modulo H is a group G/H.

Definition 1.2

Given $a \in G$. The commutator of a and b, signified by $[a, b]$, is the element
$[a, b] = aba^{-1}b^{-1} \in G$.

Definition 1.3

A group G is said to be metabelian if it has an abelian normal subgroup H, resulting in an abelian factor group G/H.

Definition 1.4

Let G be a finite group and let x_1, x_2 be elements in G. The elements x_1, x_2 are said to be conjugate if $x_2 = hx_1h^{-1}$ for some h in G. The set of all conjugate elements of x_1 is called the conjugacy classes of x_1.

Definition 1.5

Let G be a finite group the centre is a subset of elements in G that commute with every element of G, and is denoted by Z(G). In symbol the centre can be defined as:

$$Z(G) = \{a \in G : ax = xa \forall x \in G\}.$$

Definition 1.6

Suppose G is a group, two elements a and b of G are conjugate if there exists an element g in G with $gag^{-1} = b$.

Definition 1.7

Let G be a finite group. Then the conjugacy class of the element a in G is given as:

$$\mathrm{cl}(a) = \{ g \in G / xax^{-1} = g, x \in G \}.$$

3 Main Results

The main result is described in this part.

Theorem 1.

Suppose $G = D_3 \cong S_3$. Then, the group conjugate graph's energy is, $\varepsilon(\Gamma_G) = 6$.

Proof:

Let $G = D_3 \cong S_3$. Then comes the conjugate graph's adjacency matrix of A.

$\Gamma_G^C = K_2 \cup K_3$ is given in the following:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

As a result, the characteristic polynomial of A is as follows:

$$\lambda^5 - 4\lambda^3 - 2\lambda^2 + 3\lambda + 2$$

As a result, the eigenvalues are discovered to be. $\lambda = 2$, $\lambda = 1$ and $\lambda = -1$ with three repetition. Thus, the energy of the conjugate graph for D_3 is $\varepsilon(\Gamma_G) = \sum_{i=1}^{n} |\lambda_i| = 6$.

Theorem 2.

Suppose $G = D_4$. Then, the group conjugate graph's energy is, $\varepsilon(\Gamma_G) = 6$.

Proof:

Let $G = D_4$. Then comes the conjugate graph's adjacency matrix of A. $\Gamma_{D_4}^C = \{Ki_2\}_{i=1}^3$ is given in the following:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

As a result, the characteristic polynomial of A is as follows:

$$\lambda^6 - 3\lambda^4 - 3\lambda^2 - 1$$

As a result, the eigenvalues are discovered to be. $\lambda = 1$ with three repetitions and $\lambda = -1$ with three repetitions. Hence, the energy of the conjugate graph for D_4 is

$$\varepsilon(\Gamma_G) = \sum_{i=1}^{n} |\lambda_i| = 6.$$

Theorem 3.

Suppose $G = Q \times Z_2$. Then, the group conjugate graph's energy is, $\varepsilon(\Gamma_G)$ = 12.

Proof:

Let $G = Q \times Z_2$. Then comes the conjugate graph's adjacency matrix of A.

$\Gamma^{C}_{Q\times Z_2} = \{Ki_2\}_{i=1}^{6}$ is given in the following:

$$A = \begin{bmatrix} 0&1&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&1&0 \end{bmatrix}$$

As a result, the characteristic polynomial of A is as follows:

$$(-1 + \lambda^2)^6$$

As a result, the eigenvalues are discovered to be. $\lambda = 1$ with six repetition and $\lambda = -1$ with six repetition. Hence, the energy of the conjugate graph for $Q \times Z_2$is

$$\varepsilon(\Gamma_G) = \sum_{i=1}^{n} |\lambda_i| = 12.$$

Theorem 4.

Suppose G = Z_3 x Z_4. Then, the group conjugate graph's energy is, $\varepsilon(\Gamma_G)$ = 12.

Proof:

Let G = Z_3 x Z_4. Then comes the conjugate graph's adjacency matrix of A. $\Gamma^{C}_{D8} = \{Ki_2\}_{i=1}^{2} \cup \{Ki_3\}_{i=1}^{2}$ is given in the following:

$$A = \begin{bmatrix} 0&1&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&1&0&0&0&0&0 \\ 0&0&1&0&1&0&0&0&0&0 \\ 0&0&1&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&1 \\ 0&0&0&0&0&0&0&1&0&1 \\ 0&0&0&0&0&0&0&1&1&0 \end{bmatrix}$$

As a result, the characteristic polynomial of A is as follows:

$$4 + \lambda^{10} - 8\lambda^5 - 4\lambda^7 + 22\lambda^6 + 20\lambda^5 - 20\lambda^4 - 28\lambda^3 + \lambda^2 + 12\lambda$$

As a result, the eigenvalues are discovered to be. $\lambda = 2$ with 2 repetition, $\lambda = 1$ with 2 repetition and $\lambda = -1$ with six repetition. Hence, the energy of the conjugate graph for

Z_3 x Z_4 is

$$\varepsilon(\Gamma_G) = \sum_{i=1}^{n} |\lambda_i| = 12.$$

Theorem 5.

Suppose G = S_3 x Z_3. Then, the group conjugate graph's energy is, $\varepsilon(\Gamma_G)$ = 12.

Proof:

Let G = S_3 x Z_3. Then comes the conjugate graph's adjacency matrix of A. $\Gamma_G = \{Ki_4\}_{i=1}^{2}$ is given in the following:

$$A = \begin{bmatrix} 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&0 \end{bmatrix}$$

As a result, the characteristic polynomial of A is as follows:

$$(-1 + \lambda^2)^3 \; (-2 + \lambda^3 - 3\lambda)^3$$

As a result, the eigenvalues are discovered to be. $\lambda = 2$ with 3 repetition, $\lambda = 1$ with 3 repetition and $\lambda = -1$ with nine repetition. Hence, the energy of the conjugate graph S_3 x Z_3 is

$$\varepsilon(\Gamma_G) = \sum_{i=1}^{n} |\lambda_i| = 18.$$

Theorem 6.

Suppose $G = D_{10}$. Then, the group conjugate graph's energy is, $\varepsilon(\Gamma_G) = 24$.

Proof:

Let $G = D_{10}$. Then comes the conjugate graph's adjacency matrix of A.

$\Gamma^{C}_{D_{10}} = \{Ki_2\}_{i=1}^{2} \cup K_5$ is given in the following:

$$A=\begin{bmatrix}
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&1&0&1&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&1&1&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&1&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&1&0&1&1&0&0&0&0&0&0\\
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0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0
\end{bmatrix}$$

As a result, the characteristic polynomial of A is as follows:

$$(-1+\lambda^2)^4(-4+\lambda^5-10\lambda^3-20\lambda^2-15\lambda)^2$$

As a result, the eigenvalues are discovered to be, $\lambda = 4$ with 2 repetition, $\lambda = 1$ with 4 repetition and $\lambda = -1$ with twelve repetition. Hence, the energy of the conjugate graph for D_{10} is $\varepsilon(\Gamma_G) = \sum_{i=1}^{n} |\lambda_i| = 24$.

4 Conclusion

The energy of conjugate graph is determined in this section for all nonabelian metabelian of order less than 24 and the results are reduced in the table below.

Table 1.

No.	Groups	Conjugate Graph	Energy
1.	$D_6 \cong S_3$	$K_2 \cup K_3$	6
2.	$D_8 \cong Q$	$\{Ki_2\}_{i=1}^{3}$	6
3.	$D_{10} \cong A_4$	$\{Ki_2\}_{i=1}^{2} \cup K_5$	12
4.	$D_{12} \cong Z_3 \text{ x } Z_4$	$\{Ki_2\}_{i=1}^{2} \cup \{Ki_3\}_{i=1}^{2}$	12
5.	$D_{14} \cong Fr_{21} \cong Fr_{22}$	$\{Ki_2\}_{i=1}^{3} \cup K_7$	18
6.	$Q \times Z_2 \cong \text{ B } \cong K\,A_{44}$	$\{Ki_2\}_{i=1}^{6}$	12
7.	$S_3 \text{ x } Z_3$	$\{Ki_4\}_{i=1}^{2}$	12
8.	$D_{16} \cong Q \cong Quasihedra - 16$	$\{Ki_2\}_{i=1}^{3} \cup \{Ki_4\}_{i=1}^{2}$	18
9.	$D_{18} \cong (Z_3 \times Z_3) \times Z_2$	$\{Ki_2\}_{i=1}^{4} \cup K_9$	24
10.	D_{20}	$\{Ki_2\}_{i=1}^{4} \cup \{Ki_5\}_{i=1}^{2}$	24
11.	D_{22}	$\{Ki_2\}_{i=1}^{5} \cup K_{11}$	30

Competing Interests

Authors have declared that no competing interests exist.

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