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Coefficient Bounds for Bazilevic Functions Associated with Modified Sigmoid Function

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

The focus of the present paper is to obtain the sharp upper bounds of $a_2(\alpha)$ and $a_3(\alpha)$ for functions belonging to the Bazilevic class $B(\alpha,n,\gamma,\phi)$ associated with modified sigmoid function. The connection of these bounds to the celebrated Fekete-Szego functional $|a_3(\alpha)-\mu a_2^2(\alpha)|$ follows as simple consequence.

Keywords: Analytic function; univalent function; Bazilevic function; sigmoid function.

1 Introduction

Let A denote the class of all functions of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k \qquad z\in D \tag{1.1}$$

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which are analytic in the open unit disk $D=\{z;|z|<1\}$ and normalized by

$f(0)=f^{1}(0)-1=0.$ Also, let S denote the subclass of A which are normalized and univalent in D.

In 1983, Sălăgean [1] introduced and studied the following differential operator:

$$\begin{aligned} &D^0 f(z)= f(z)\\ &D^1 f(z)=D(D^0 f(z))=z\, f'(z)\\ &D^n f(z)=D(D^{n-1} f(z))=z(D^{n-1} f(z))'. \end{aligned} \tag{1.2}$$

From equation (1.1), we can write that

$$f(z)^{\alpha}=\left(z+\sum_{k=2}^{\infty} a_k z^k\right)^{\alpha} \tag{1.3}$$

Expanding equation (1.3) binomially, then

$$f(z)^{\alpha}=z^{\alpha}+\sum_{k=2}^{\infty} a_k(\alpha)\, z^{\alpha+k-1} \tag{1.4}$$

where $\alpha>0$ that is, α is real. Using (1.2) and (1.4), then

$$D^n f(z)^{\alpha}=\alpha^n z^{\alpha}+\sum_{k=2}^{\infty}(\alpha+k-1)^n a_k(\alpha) z^{\alpha+k-1} \qquad \alpha>0,\ \ z\in D \tag{1.5}$$

is obtained. Now, consider the following function:

$$\left\{\frac{\alpha}{1+\varepsilon^2}\int_0^z \frac{p(v)-i\varepsilon}{V^{\left(1+\frac{i\alpha\varepsilon}{(1+\varepsilon^2)}\right)}}\, g(v)^{\frac{\alpha}{1+\varepsilon^2}}\, dv\right\}^{\frac{1+i\varepsilon}{\alpha}} \tag{1.6}$$

where $\alpha>0$ (α are real), $p\in P$ and $g\in\Psi^{*}$. The genesis of the study of the function given above in (1.6) is the discovery in 1955 by a Russian Mathematician called Bazilevic [2]. The family of functions (1.6) became known as Bazilevic functions and is usually denoted by $B(\alpha,\varepsilon)$. Very little is known about this family of functions defined in (1.6), except that, he Bazilevic showed that each function $f\in B(\alpha,\varepsilon)$ is univalent in D. However, by simplifying (1.6) it is quite possible to understand and investigate the family better. It should be noted that with special choices of parameters α,ε and the function $g(z)$, the family $B(\alpha,\varepsilon)$ cracks down to some well-known subclasses of univalent functions (see [3] for details). For instance, if we let $\varepsilon=0$ then (1.6) immediately yields

$$f(z)=\left\{\alpha\int_0^z \frac{p(v)}{V} g(v)^{\alpha}\, dv\right\}^{\frac{1}{\alpha}}. \tag{1.7}$$

By differentiating equation (1.7) we have

$$\frac{z\, f'(z)\, f(z)^{\alpha-1}}{g(z)^{\alpha}} = p(z), \qquad z\in D \tag{1.8}$$

or equivalently

$$\Re e\left\{\frac{z\, f'(z)\, f(z)^{\alpha-1}}{g(z)^{\alpha}}\right\} > 0, \qquad z\in D \tag{1.9}$$

The subclass of Bazilevic functions satisfying equation (1.8) are called Bazilevic functions of type α and are denoted by $B(\alpha)$ (see Singh, [4]). In 1973, Noonan [5] gave a plausible description of functions of the class $B(\alpha)$ as those functions in Ψ for which each $r>1$, and the tangent to the curve $U_{\alpha}(r)=\left\{\varepsilon f(re^{i\theta})^{\alpha},\, 0\leq\theta<2\pi\right\}$ never turns back on itself as much as π radian. If $\alpha=1$, the class $B(\alpha)$ reduces to the family of close-to-convex functions; that is,

$$\Re e\left\{\frac{zf'}{g(z)}\right\} > 0 \qquad z\in D. \tag{1.10}$$

If we decide to choose $g(z)=f(z)$ in inequality (1.10), we have

$$\Re e\left\{\frac{zf'}{f(z)}\right\} > 0 \qquad z\in D.$$

This implies that $f(z)$ is starlike. Furthermore, if one replaces $f(z)$ by $zf'(z)$, then

$$\Re e\left\{1+\frac{zf''}{f'}\right\} > 0 \qquad z\in D.$$

This shows that $f(z)$ is convex. Moreover, if $g(z)=z$ in inequality (1.9), then the family of $B_1(\alpha)$ of functions satisfying

$$\Re e\left\{\frac{z\, f'(z)\, f(z)^{\alpha-1}}{z^{\alpha}}\right\} > 0, \qquad z\in D \tag{1.11}$$

is obtained.

In 1992, Abdulhalim [6] introduced a generalization of (1.11) such that

$$\Re e\left\{\frac{D^n f(z)^\alpha}{z^\alpha}\right\} > 0, \qquad \alpha > 0, \; z \in D. \tag{1.12}$$

where the parameter $\alpha > 0$ and the operator D^n is the famous Sălăgean derivative operator [1] defined in (1.2). Further in 1994, Opoola [7] studied a more generalization of (1.12) and denoted it by $T_n^\alpha(\gamma)$ (Bazilevic class of order gamma) such that

$$\Re e\left\{\frac{D^n f(z)^\alpha}{z^\alpha}\right\} > \gamma, \quad \gamma \geq 0, \; \alpha > 0, \quad z \in D. \tag{1.13}$$

Recently, a little modification was made to (1.13) such that

$$\Re e\left\{\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}\right\} > 0 \quad \alpha > 0, \; n \in N_0, \; z \in D. \tag{1.14}$$

Here, it is noted that

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} = 1 + \sum_{k=2}^{\infty}\left(\frac{\alpha + k - 1}{\alpha}\right)^n a_k(\alpha) z^{k-1} \tag{1.15}$$

where for convenience we denote the class of functions in (1.15) by $B(\alpha, n)$.

Now, the theory of both the analytic functions and special functions (such as sigmoid function) are of great importance in addressing many physical problems such as in heat conduction and aerodynamic to mention but few.

It is generally believed that activation function is an information process that is inspired by the way nervous system like brain, process information. It is composed of large number of highly interconnected processing element (neurons) working to solve a specific assignment. This function works in similar way the brain does. The human brain can be regarded as an information-processing entity. It receives information from the external environment through the sense and processes them to form internal models of external phenomena. In particular, the brain is capable of redressing these models to suit new situations and then make reliable decisions.

The most widely used sigmoid function is the logistic activation function which has a lower bound of zero (0) and upper bound of one (1). It means that the function value (or the output) range is [0, 1].

Many sigmoid functions have power series expansion which alternate in sign while some have inverse with hypergeometric series expansion. They can be evaluated differently especially by truncated series expansion. The logistic sigmoid function is defined as

$$g(z) = \frac{1}{1 + e^{-z}} = \frac{1}{2} + \frac{1}{4}z - \frac{1}{48}z^3 + \frac{1}{480}z^5 - \ldots \tag{1.16}$$

and has the following properties:

(i) It outputs real number between 0 and 1
(ii) It maps a very large input domain to a small range of outputs
(iii) It never loses information because it is a one-to-one function
(iv) It increases monotonically.

In view of the above properties sigmoid function is highly useful in geometric Function Theory (See [8,9] for more detail).

However, the investigation of Fadipe-Joseph et al. [8] on the logistic sigmoid function has stirred the interest of both young and old researchers in the field of geometric function theory with several interesting results authenticated diversely in literatures. Motivated by the work of Fadipe-Joseph et al. [8], Oladipo and Gbolagade [9], the author here wishes to investigate the coefficient bounds for certain class of analytic functions involving modified sigmoid function in the unit disk.

2 Coefficient Bounds

Let P (Caratheodory functions) be the family of all functions p analytic in D for which $p(0)=1$, $\Re\{p(z)\}>0$ and

$$p(z)=1+p_1z+p_2z^2+\ldots \qquad z\in D \tag{2.1}$$

in the unit disk D (see [10]).

Lemma 2.1 [11]: Let $p\in P$. Then $|p_k|\leq 2,\ k=1,2,3,4,\ldots$. Equality is attained by the moebius function

$$L_0=\frac{1+z}{1-z}.$$

Lemma 2.2[8,12]: Let g be sigmoid function of the form (1.16). Then, let $\phi(z)=2g(z)$ such that

$$\phi(z)=1+\sum_{k=1}^{\infty}\frac{(-1)^k}{2^k}\left(\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}z^m\right)^k. \tag{2.2}$$

Then $\phi(z)\in P$, $|z|<1$ where P is the class of Caratheodory functions and $\phi(z)$ denotes the celebrated modified sigmoid function.

Lemma 2.3 [8,12]: Let

$$\phi(z)=1+\sum_{k=1}^{\infty}\frac{(-1)^k}{2^k}\left(\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}z^m\right)^k \tag{2.3}$$

Then, $|\phi(z)|<2$.

Now, let $f(z)^{\alpha}$ of the form (1.4) belong to $B(\alpha,n)$. Then, for $\alpha>0;\ 0\leq\gamma\leq 1;\ n\in N_0=N\cup\{0\}$

and $-1 \le B < A \le 1$

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} \prec \frac{1+Az}{1+Bz}. \tag{2.4}$$

Hence, by the definition of subordination, it follows that $f(z)^\alpha \in B(\alpha,n,\gamma,\phi)$ if and only if

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} = \frac{1+Az}{1+Bz} = (1-\gamma)\,p(z) + \gamma\,\phi(z) \tag{2.5}$$

where $\phi(z) = 2\,g(z)$ and $g(z)$ is as defined in (1.16).

Next is the coefficient bounds for functions in the Bazilevic class $B(\alpha,n,\gamma,\phi)$.

Theorem 2.4: Suppose that $f(z)^\alpha \in B(\alpha,n,\gamma,\phi)$. Then, for $\alpha > 0;\ 0 \le \gamma \le 1;\ n \in N_0$

$$|a_2(\alpha)| \le \frac{\alpha^n(4-3\gamma)}{2(\alpha+1)^n}, \quad |a_3(\alpha)| \le \frac{2\alpha^n(1-\gamma)}{(\alpha+2)^n}, \quad |a_4(\alpha)| \le \frac{\alpha^n(48-47\gamma)}{24(\alpha+3)^n},$$
$$|a_5(\alpha)| \le \frac{2\alpha^n(1-\gamma)}{(\alpha+4)^n}, \quad |a_6(\alpha)| \le \frac{\alpha^n(480-479\gamma)}{240(\alpha+5)^n}.$$

Proof: Let $f(z)^\alpha \in B(\alpha,n,\gamma,\phi)$. Then, there exists $p, \phi(z) \in P$ (class of caratheodory functions) such that

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} = (1-\gamma)\,p(z) + \gamma\,\phi(z). \tag{2.6}$$

where

$$\phi(z) = 1 + \tfrac{1}{2}z - \tfrac{1}{24}z^3 + \tfrac{1}{240}z^5 - \tfrac{1}{64}z^6 + \dots. \tag{2.7}$$

Then,

$$\begin{aligned} &1+\alpha_{n,2}\,a_2(\alpha)z+\alpha_{n,3}\,a_3(\alpha)z^2+\alpha_{n,4}\,a_4(\alpha)z^3+\alpha_{n,4}\,a_4(\alpha)z^4+\alpha_{n,5}\,a_5(\alpha)z^5+\dots \\ &=1+\left(p_1-\frac{\gamma}{2}(2\,p_1-1)\right)z+(1-\gamma)\,p_2z^2+\left(p_3-\frac{\gamma}{24}(24\,p_3-1)\right)z^3+(1-\gamma)\,p_4\,z^4 \\ &+\left(p_5-\frac{\gamma}{240}(240\,p_5-1)\right)z^5+\dots. \end{aligned} \tag{2.8}$$

Equating the coefficient of the like powers of z, z^2, z^3, z^4 and z^5 in (2.8) above, then

$$a_2(\alpha) = \frac{\alpha^n[2\,p_1-\gamma(2\,p_1-1)]}{2(\alpha+1)^n} \tag{2.9}$$

$$a_3(\alpha) = \frac{\alpha^n (1-\gamma) p_2}{(\alpha+2)^n} \tag{2.10}$$

$$a_4(\alpha) = \frac{\alpha^n [24\, p_3 - \gamma(24\, p_3 - 1)]}{24\,(\alpha+3)^n} \tag{2.11}$$

$$a_5(\alpha) = \frac{\alpha^n (1-\gamma) p_4}{(\alpha+4)^n} \tag{2.12}$$

$$a_6(\alpha) = \frac{\alpha^n [240\, p_5 - \gamma(240\, p_5 - 1)]}{240\,(\alpha+5)^n} \tag{2.13}$$

Applying Lemma 2.1, then we obtain the desired results. This ends the proof of Theorem 2.4.

Corollary 2.5: Suppose that $f(z)^\alpha \in B(1, n, \gamma, \phi)$. Then, for $\alpha > 0;\ 0 \le \gamma \le 1;\ n \in N_0$

$$|a_2(1)| \le \frac{(4-3\gamma)}{2(2)^n}, \quad |a_3(1)| \le \frac{2(1-\gamma)}{(3)^n}, \quad |a_4(1)| \le \frac{(48-47\gamma)}{24\,(4)^n},$$
$$|a_5(1)| \le \frac{2\,(1-\gamma)}{(5)^n}, \quad |a_6(1)| \le \frac{(480-479\gamma)}{240\,(6)^n}.$$

Corollary 2.6: Suppose that $f(z)^\alpha \in B(1, 0, \gamma, \phi)$. Then, for $\alpha > 0;\ 0 \le \gamma \le 1;\ n \in N_0$

$$|a_2(1)| \le \frac{(4-3\gamma)}{2}, \quad |a_3(1)| \le 2(1-\gamma), \quad |a_4(1)| \le \frac{(48-47\gamma)}{24},$$
$$|a_5(1)| \le 2\,(1-\gamma), \quad |a_6(1)| \le \frac{(480-479\gamma)}{240}.$$

Corollary 2.7: Suppose that $f(z)^\alpha \in B(1, 1, \gamma, \phi)$. Then, for $\alpha > 0;\ 0 \le \gamma \le 1;\ n \in N_0$

$$|a_2(1)| \le \frac{(4-3\gamma)}{4}, \quad |a_3(1)| \le \frac{2(1-\gamma)}{3}, \quad |a_4(1)| \le \frac{(48-47\gamma)}{96},$$
$$|a_5(1)| \le \frac{2\,(1-\gamma)}{5}, \quad |a_6(1)| \le \frac{(480-479\gamma)}{1440}.$$

Corollary 2.8: Suppose that $f(z)^\alpha \in B(1, 0, 0, \phi)$. Then, for $\alpha > 0;\ 0 \le \gamma \le 1;\ n \in N_0$

$$|a_2(1)| \le 2, \quad |a_3(1)| \le 2, \quad |a_4(1)| \le 2, \quad |a_5(1)| \le 2, \quad |a_6(1)| \le 2$$

and in general

$$|a_k| \le 2, \quad k = 2, 3, 4, \ .$$

In the recent time, Fekete-Szego inequality has been one of the fascinating problems beckoning the attention of both young and old researchers in the field of complex analysis. They have succeeded not only in obtaining sharp bounds for the first two initial coefficients $|a_2|$ and $|a_3|$ for various subclasses of *S*, but also in establishing a close link or connection between these coefficients and the functional $\left|a_3 - \mu a_2^2\right|$ (see [13, 12, 14] among others). Here, the author uses the values of a_2 and a_3 obtained in (2.9) and (2.10) respectively, to prove the Fekete-Szego result for the function class $B(\alpha, n, \gamma, \phi)$ involving modified sigmoid function.

Theorem 2.9: Suppose that $f(z)^\alpha \in B(\alpha, n, \gamma, \phi)$. Then, for $\alpha > 0;\ 0 \le \gamma \le 1;\ n \in N_0$.

$$\left|a_3(\alpha) - \mu a_2^2(\alpha)\right| \le \frac{\alpha^n}{4(\alpha+1)^{2n}(\alpha+2)^n}\left|8(\alpha+1)^{2n}(1-\gamma) - \mu\alpha^n(\alpha+2)^n\ (9\gamma^2 - 12\gamma + 16)\right|.$$

Proof: Using (2.9) and (2.10) with Lemma 2.1, the proof is immediate.

Corollary 2.10: Let $f(z)^\alpha \in B(1, n, \gamma, \phi)$. Then, for $\alpha > 0;\ 0 \le \gamma \le 1;\ n \in N_0$

$$\left|a_3(1) - \mu a_2^2(1)\right| \le \frac{1}{4(2)^{2n}(3)^n}\left|8(2)^{2n}(1-\gamma) - \mu\ (3)^n(9\gamma^2 - 12\gamma + 16)\right|.$$

Corollary 2.11: Let $f(z)^\alpha \in B(1, 1, \gamma, \phi)$. Then, for $\alpha > 0;\ 0 \le \gamma \le 1;\ n \in N_0$

$$\left|a_3(1) - \mu a_2^2(1)\right| \le \frac{1}{48}\left|32(1-\gamma) - 3\mu\ (9\gamma^2 - 12\gamma + 16)\right|.$$

Corollary 2.12: Let $f(z)^\alpha \in B(1, 0, \gamma, \phi)$. Then, for $\alpha > 0;\ 0 \le \gamma \le 1;\ n \in N_0$

$$\left|a_3(1) - \mu a_2^2(1)\right| \le \frac{1}{4}\left|8\ (1-\gamma) - \mu\ \ (9\gamma^2 - 12\gamma + 16)\right|.$$

Corollary 2.13: Let $f(z)^\alpha \in B(1, 0, 0, \phi)$. Then, for $\alpha > 0;\ 0 \le \gamma \le 1;\ n \in N_0$

$$\left|a_3(1) - \mu a_2^2(1)\right| \le 2\left|1 - 2\mu\right|.$$

Corollary 2.14: Let $f(z)^\alpha \in B(1, 1, 0, \phi)$. Then, for $\alpha > 0;\ 0 \le \gamma \le 1;\ n \in N_0$

$$\left|a_3(1) - \mu a_2^2(1)\right| \le \frac{1}{3}\left|2 - 3\mu\right|.$$

Final Remark:

If μ=1 in corollary 2.13 and corollary 2.14 respectively, then

$$\left|a_3(1) - \mu a_2^2(1)\right| \le 2.$$

and

$$\left|a_3(1)-\mu a_2^2(1)\right| \le \frac{1}{3}.$$

For some results on Fekete-Szego problem see [13,14] among others.

Theorem 2.15: Suppose that $f(z)^\alpha \in B(\alpha,n,\gamma,\phi)$. Then, for $\alpha > 0;\ 0 \le \gamma \le 1;\ n \in N_0$.

$$\left|a_2(\alpha)\,a_4(\alpha)-\mu\,a_3^2(\alpha)\right| \le \frac{\alpha^{2n}}{4(\alpha+1)^n(\alpha+2)^{2n}(\alpha+3)^n}\left|\begin{matrix}(\alpha+2)^{2n}(4-3\gamma)(48-47\gamma)\\ -192(\alpha+1)^n(\alpha+3)^n(1-\gamma)^2\end{matrix}\right|.$$

Proof: Using (2.9), (2.10) and (2.11) with Lemma 2.1, the proof is immediate.

3 Conclusion

By substituting zero (0) for the value of γ in all the results obtained in this paper, then we would be having the results associated with the class p(z) of Caratheodory functions defined in (2.1) alone while by letting $\gamma = 1$, then all the results obtained would be associated with the modified sigmoid function $\phi(z)$ defined in (2.7) alone.

Competing Interests

Author has declared that no competing interests exist.

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