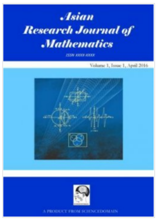

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On Generalization of a Reverse Hilbert's Type Inequality

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Authors' contributions

This work was carried out in collaboration between both authors. Author Xi designed the study, performed the analysis, wrote the protocol and wrote the first draft of the manuscript. Author Zhang proof read the manuscript and effected the corrections. Both authors read and approved the final manuscript.

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Abstract

We establish an inequality of a weight coefficient by introducing a parameter λ and using the Euler-Maclaurin expansion. Using this inequality, we derive a reverse of the Hilbert's type inequality. As an applications, an equivalent form is obtained.

Keywords: Hilbert's inequality; weight coefficient; Euler-Maclaurin expansion; reverse.

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1 Introduction and Main Result

If $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n \geq 0$, $b_n \geq 0$, and $0 < \sum_{n=1}^{\infty} a_n^p < \infty$, $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{\sum_{n=1}^{\infty} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} b_n^q\right\}^{\frac{1}{q}}, \tag{1.1}$$

and

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m,n\}} < pq \left\{\sum_{n=1}^{\infty} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} b_n^q\right\}^{\frac{1}{q}}, \tag{1.2}$$

where the constant $\frac{\pi}{\sin\frac{\pi}{p}}$ and pq is best possible for each inequality respectively. Inequality (1.1) is Hardy-Hilbert's inequality. Inequality (1.2) is a Hilbert's type inequality [1].

In [2], Yang gave a reinforcement of inequality (1.1):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{m^2+n^2} < \frac{\pi}{2} \left\{\sum_{n=1}^{\infty} \frac{1}{n} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} \frac{1}{n} b_n^q\right\}^{\frac{1}{q}}, \tag{1.3}$$

In [3], [4] and [5], Krnic, Pecaric and Yang gave some generalization and reinforcement of inequality (1.1). In [6], Kuang and Debnath gave a reinforcement of inequality (1.2):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m,n\}} < \left\{\sum_{n=1}^{\infty} [pq - G(p,n)] a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} [pq - G(q,n)] b_n^q\right\}^{\frac{1}{q}}, \tag{1.4}$$

where $G(r,n) = \frac{r + \frac{1}{3r} - \frac{4}{3}}{(2n+1)^{\frac{1}{r}}} > 0$ $(r = p, q)$.

In [7] and [8], Xi gave a generalization and reinforcement of inequalities (1.2) and (1.4):

$$\begin{aligned} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m^\lambda, n^\lambda\}} \quad < & \quad \left\{\sum_{n=1}^{\infty} \left[\kappa\left(\lambda\right) - \frac{1}{3qn^{\frac{q+\lambda-2}{q}}}\right] n^{1-\lambda} a_n^p\right\}^{\frac{1}{p}} \\ & \times \left\{\sum_{n=1}^{\infty} \left[\kappa\left(\lambda\right) - \frac{1}{3pn^{\frac{p+\lambda-2}{p}}}\right] n^{1-\lambda} b_n^q\right\}^{\frac{1}{q}}, \end{aligned} \tag{1.5}$$

where $\kappa(\lambda) = \frac{p\,q\lambda}{(p+\lambda-2)(q+\lambda-2)} > 0$, $2 - \min\{p,q\} < \lambda \leq 2$.

$$\begin{aligned} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m^\lambda + A, n^\lambda + B\}} < & \left\{\sum_{n=1}^{\infty} \left[\kappa(\lambda) - \frac{1}{n^{\frac{q+\lambda-2}{q}}} \left(\frac{1}{3q} - \frac{B}{1+B}\right)\right] n^{1-\lambda} a_n^p\right\}^{\frac{1}{p}} \\ & \times \left\{\sum_{n=1}^{\infty} \left[\kappa(\lambda) - \frac{1}{n^{\frac{p+\lambda-2}{p}}} \left(\frac{1}{3p} - \frac{A}{1+A}\right)\right] n^{1-\lambda} b_n^q\right\}^{\frac{1}{q}}, \end{aligned} \tag{1.6}$$

For the reverse Hardy-Hilbert's inequality, Yang [9] gave a reverse form of inequalities (1.3). In [10] , Xi and Wang gave a reverse Hilbert's type inequality:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^2,n^2\}} \quad > \quad 2\left[\sum_{n=1}^{\infty}\left(1-\frac{1}{2n}\right)\frac{1}{n}a_n^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}\frac{1}{n}b_n^q\right]^{\frac{1}{q}}. \tag{1.7}$$

In this paper, by introducing a parameter λ and using the Euler-Maclaurin expansion, we establish an inequality of a weight coefficient. Using this inequality, we derive a reverse of the Hilbert's type inequality (1.5) and a generalization of inequalities (1.7). The main result of this paper is the following inequality.

Theorem 1. *If* $0 < p < 1$, $\frac{1}{p}+\frac{1}{q} = 1, \lambda > 1$, $a_n \geq 0$, $b_n \geq 0$ *and* $0 < \sum\limits_{n=1}^{\infty} a_n^p < \infty$, $0 < \sum\limits_{n=1}^{\infty} \frac{1}{n}b_n^q < \infty$, *then*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda,n^\lambda\}} > \frac{\lambda}{\lambda-1}\left\{\sum_{n=1}^{\infty}\left[1-\frac{(\lambda-1)(\lambda+2)}{4\lambda n}\right]\frac{a_n^p}{n^{\lambda-1}}\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\frac{b_n^p}{n^{\lambda-1}}\right\}^{\frac{1}{q}}. \tag{1.8}$$

Before we give the proof of the theorem, we need the following expression of the Euler-Maclaurin (see [11])

$$\sum_{k=n+1}^{m} f(k) = \int_n^m f(x)dx + \frac{1}{2}[f(m)-f(n)] + \int_n^m P_1(x)f'(x)dx, \tag{1.9}$$

where $f(x) \in C^1[0,\ \infty)$, $m, n \in N_0(m > n)$, N_0 is the set of non-negative integers, $P_i(x)(i = 1, 2, \cdots)$ are Bernoulli function ($P_1(x) = x - [x] - \frac{1}{2}$). When $\sum_{k=n}^{\infty} f(k)$, $\int_n^{\infty} f(x)dx$ are convergences, we have

$$\sum_{k=n}^{\infty} f(k) = \int_n^{\infty} f(x)dx + \frac{1}{2}f(n) + \int_n^{\infty} P_1(x)f'(x)dx, \tag{1.10}$$

and(see [9])

$$\int_n^{\infty} P_1(x)g(x)dx = -\frac{1}{8}g(n)\varepsilon(0 < \varepsilon < 1), \tag{1.11}$$

where $g(x) \in C^1[0,\ \infty), g'(x) < 0(\text{or } g'(x) > 0), x \in [n,\infty), g(\infty) = 0$.

2 A Lemma

Lemma 1. *Let N be the set of positive integers. The weight coefficient $\omega(n)$ is defined by*

$$\omega(n,\lambda) = \sum_{k=1}^{\infty}\frac{1}{max\{k^\lambda,n^\lambda\}}, \qquad n \in N, \lambda > 1.$$

Then we have

$$\frac{\lambda}{(\lambda-1)n^{\lambda-1}}\left[1-\frac{(\lambda-1)(\lambda+2)}{4\lambda n}\right] < \omega(n,\lambda) < \frac{\lambda}{(\lambda-1)n^{\lambda-1}}. \tag{2.1}$$

Proof. If $n \in N$, let $f(x) = \frac{1}{max\{x^\lambda,n^\lambda\}}$, $\quad x \in [0,\quad \infty)$, we have

$$f(x) = \frac{1}{max\{x^\lambda,n^\lambda\}} = \begin{cases} \frac{1}{n^\lambda}, & x < n, \\ \frac{1}{x^\lambda}, & x \geq n, \end{cases}$$

and

$$f'(x) = \begin{cases} 0, & x < n, \\ -\frac{\lambda}{x^{\lambda+1}}, & x \geq n. \end{cases}$$

By (1.10), we obtain

$$\omega(n,\lambda) = \int_1^\infty f(x)dx + \frac{1}{2}f(1) + \int_1^\infty P_1(x)f'(x)dx$$
$$= \frac{1}{n^{\lambda-1}} - \frac{1}{n^\lambda} + \frac{1}{(\lambda-1)n^{\lambda-1}} + \frac{1}{2n^\lambda} - \int_n^\infty P_1(x)\frac{\lambda}{x^{\lambda+1}}dx.$$

By (1.11), we have

$$-\frac{\lambda}{8n^{\lambda+1}} < \int_n^\infty P_1(x)\frac{\lambda}{x^{\lambda+1}}dx < 0.$$

Since we find

$$\frac{\lambda}{(\lambda-1)n^{\lambda-1}} - \frac{1}{2n^\lambda} - \frac{\lambda}{8n^{\lambda+1}} < \omega(n,\lambda) < \frac{\lambda}{(\lambda-1)n^{\lambda-1}} - \frac{1}{2n^\lambda},$$

$$\frac{\lambda}{(\lambda-1)n^{\lambda-1}}\left[1 - \frac{(\lambda-1)(\lambda+2)}{4\lambda n}\right] < \omega(n,\lambda) < \frac{\lambda}{(\lambda-1)n^{\lambda-1}}.$$

Then we have (2.1). The lemma is proved.

3 The Proof and Application of Theorem

In this section, we use (2.1) to prove Theorem 1. As applications, an equivalent form is obtained.

Proof. By the reverse Hölder 's inequality [12], we have

$$\sum_{n=1}^\infty\sum_{m=1}^\infty \frac{a_m b_n}{\max\{m^\lambda, n^\lambda\}} = \sum_{n=1}^\infty\sum_{m=1}^\infty \left[\frac{a_m}{(\max\{m^\lambda,n^\lambda\})^{\frac{1}{p}}}\right]\left[\frac{b_n}{(\max\{m^\lambda,n^\lambda\})^{\frac{1}{q}}}\right]$$
$$\geq \left\{\sum_{n=1}^\infty\sum_{m=1}^\infty\left[\frac{a_m^p}{\max\{m^\lambda,n^\lambda\}}\right]\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^\infty\sum_{m=1}^\infty\left[\frac{b_n^q}{\max\{m^\lambda,n^\lambda\}}\right]\right\}^{\frac{1}{q}}$$
$$= \left\{\sum_{m=1}^\infty \omega(m,\lambda)a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^\infty \omega(n,\lambda)b_n^q\right\}^{\frac{1}{q}}.$$

Since $0 < p < 1$ and $q < 0$, remove by (2.1), we obtain (1.8). Theorem 1 is proved.

Theorem 2. *If* $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 1$, $a_n \geq 0$, $b_n \geq 0$ *and* $0 < \sum\limits_{n=1}^\infty \frac{a_n^p}{n^{\lambda-1}} < \infty$, *then*

$$\sum_{n=1}^\infty\left(\frac{1}{n^{\lambda-1}}\right)^{1-p}\left[\sum_{m=1}^\infty\frac{a_m}{\max\{m^\lambda,n^\lambda\}}\right]^p > \left(\frac{\lambda}{\lambda-1}\right)^p\sum_{n=1}^\infty\left[1-\frac{(\lambda-1)(\lambda+2)}{4\lambda n}\right]\frac{a_n^p}{n^{\lambda-1}}. \quad (3.1)$$

Inequalities (3.1) *and* (1.8) *are equivalent.*

Proof. Let

$$b_n = \left(\frac{1}{n^{\lambda-1}}\right)^{1-p}\left[\sum_{m=1}^\infty\frac{a_m}{\max\{m^\lambda,n^\lambda\}}\right]^{p-1}, \qquad n \in N.$$

By (1.8), we have

$$\begin{aligned}\left\{\sum_{n=1}^{\infty}\frac{b_n^q}{n^{\lambda-1}}\right\}^p &= \left\{\sum_{n=1}^{\infty}\left(\frac{1}{n^{\lambda-1}}\right)^{1-p}\left(\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^\lambda,n^\lambda\}}\right)^p\right\}^p \\ &= \left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda,n^\lambda\}}\right\}^p \\ &\geq \left(\frac{\lambda}{\lambda-1}\right)^p\sum_{n=1}^{\infty}\left[1-\frac{(\lambda-1)(\lambda+2)}{4\lambda n}\right]\frac{a_n^p}{n^{\lambda-1}}\left\{\sum_{n=1}^{\infty}\frac{b_n^q}{n^{\lambda-1}}\right\}^{p-1}.\end{aligned}$$

Then we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{b_n^q}{n^{\lambda-1}} &= \sum_{n=1}^{\infty}\left(\frac{1}{n^{\lambda-1}}\right)^{1-p}\left[\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^\lambda,n^\lambda\}}\right]^p \\ &\geq \left(\frac{\lambda}{\lambda-1}\right)^p\sum_{n=1}^{\infty}\left[1-\frac{(\lambda-1)(\lambda+2)}{4\lambda n}\right]\frac{a_n^p}{n^{\lambda-1}}. \qquad (3.2)\end{aligned}$$

If $\sum_{n=1}^{\infty}\frac{b_n^q}{n^{\lambda-1}}=\infty$, then in view of

$$0<\sum_{n=1}^{\infty}\left[1-\frac{(\lambda-1)(\lambda+2)}{4\lambda n}\right]\frac{a_n^p}{n^{\lambda-1}}\leq\sum_{n=1}^{\infty}\frac{a_n^p}{n^{\lambda-1}}<\infty$$

and (4.1), we have

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^{\lambda-1}}\right)^{1-p}\left[\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^\lambda,n^\lambda\}}\right]^p>\left(\frac{\lambda}{\lambda-1}\right)^p\sum_{n=1}^{\infty}\left[1-\frac{(\lambda-1)(\lambda+2)}{4\lambda n}\right]\frac{a_n^p}{n^{\lambda-1}};$$

if $0<\sum_{n=1}^{\infty}\frac{b_n^q}{n^{\lambda-1}}<\infty$, then by (1.8), we find

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^{\lambda-1}}\right)^{1-p}\left[\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^\lambda,n^\lambda\}}\right]^p>\left(\frac{\lambda}{\lambda-1}\right)^p\sum_{n=1}^{\infty}\left[1-\frac{(\lambda-1)(\lambda+2)}{4\lambda n}\right]\frac{a_n^p}{n^{\lambda-1}}.$$

Hence we obtain (3.1).

On the other-hand, by the reverse Hölder 's inequality [12], we have

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda,n^\lambda\}} &= \sum_{n=1}^{\infty}\left[\left(\frac{1}{n^{\lambda-1}}\right)^{-\frac{1}{q}}\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^\lambda,n^\lambda\}}\right]\left[\left(\frac{1}{n^{\lambda-1}}\right)^{\frac{1}{q}}b_n\right] \\ &\geq \left[\sum_{n=1}^{\infty}\left(\frac{1}{n^{\lambda-1}}\right)^{1-p}\left(\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^\lambda,n^\lambda\}}\right)^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}\frac{b_n^q}{n^{\lambda-1}}\right]^{\frac{1}{q}}.\end{aligned}$$

From (3.1), it follows that

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda,n^\lambda\}} > \frac{\lambda}{\lambda-1}\left\{\sum_{n=1}^{\infty}\left[1-\frac{(\lambda-1)(\lambda+2)}{4\lambda n}\right]\frac{a_n^p}{n^{\lambda-1}}\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\frac{b_n^p}{n^{\lambda-1}}\right\}^{\frac{1}{q}}.$$

Then, (3.1) and (1.8) are equivalent. Theorem 2 is proved.

4 Conclusion

Inequality (1.2) is Hilberts type inequality, and is important in analysis and its application. Kuang gave a strengthened version of (1.2); Yang considered a refinement of another Hilberts type inequality. For the reverse Hardy-Hilbert's inequality, Yang gave a reverse form of inequalities. Xi and Wang gave a reverse Hilbert's type inequality:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^2, n^2\}} \quad > \quad 2\left[\sum_{n=1}^{\infty}\left(1-\frac{1}{2n}\right)\frac{1}{n}a_n^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}\frac{1}{n}b_n^q\right]^{\frac{1}{q}}. \tag{4.1}$$

By introducing a parameter λ and using the Euler-Maclaurin expansion, we establish an inequality of a weight coefficient. Using this inequality, we derive a reverse of the Hilbert's type inequality and is a generalization of inequalities (4.1).

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Hardy GH, Littlewood JE, Polya G. Inequalities. Cambridge Univ. Press; 1934.

[2] Yang B. On an extension of Hardy-Hilbert's inequality with a parameter. Mathematics In Practice and Theory. 2006;36(4):226-231.

[3] Krnic M, Pecaric J. General Hilbert's and Hardy's Inequalities. Math. Inequal. Appl. 2005;8(1):29-51.

[4] Yang B. On best extensions of Hardy-Hilbert's inequality with two parameters. J. Inequal. Pure Appl. Math. 2005;6(3):Article 81.

[5] Yang B. A new Hilbert-type inequality. Bull. Belg. Math. Soc. 2006;13:479-487.

[6] Kuang J, Debnath L. On new generalizations of Hilbert's inequality and their applications. J. Math. Anal. Appl. 2000;245:248-265.

[7] Xi G. On generalizations and reinforcements of a Hilbert's type inequality. Rocky Mountain J. Mathematics. 2012;42(1):329-337.

[8] Xi G. A generalization of the Hilbert's type inequality. Mathematical Inequalities & Applications. 2015;18(4):1501-1510.

[9] Yang B. A reverse of the Hardy-Hilbert's type inequality. Mathematics In Practice and Theory. 2006;36(11):207-212.

[10] Xi G, Xi Y, Wang L. A reverse Hilbert's type inequality. Applied Mechanics and Materials. 2015;701-702:106-109.

[11] Yang B, Zhu J. Inequality on the Hurwitz Zeta-function restricted to the axis of positive reals. Acta Scientiarum Naturalium Universitatis Sunyatseni. 1997;36(2):30-35.

[12] Kuang J. Applied Inequalities. Sandong Science and Technology Press, Jinan; 2004.

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