Asian Research Journal of Mathematics

5(4): 1-12, 2017; Article no.ARJOM.35466
ISSN: 2456-477X

Unconditional Integrability with Application to Ergodic Theorem

Mangatiana A. Robdera[1*] and Dintle N. Kagiso[1]

[1]*Department of Mathematics, University of Botswana, Private Bag 0022, Gaborone, Botswana.*

Authors' contributions

This work was carried out in collaboration between the two authors. Author MAR wrote the first draft of manuscript. Both authors read and approved the final manuscript.

Article Information

DOI: 10.9734/ARJOM/2017/35466
Editor(s):
(1) Sheng Zhang, School of Mathematics and Physics, Bohai University, Jinzhou, China.
Reviewers:
(1) Choonkil Park, Hanyang University, Republic of Korea.
(2) Sie Long Kek, Universiti Tun Hussein Onn Malaysia, Malaysia.
Complete Peer review History: http://www.sciencedomain.org/review-history/20405

Received: 14th July 2017
Accepted: 3rd August 2017
Published: 7th August 2017

Original Research Article

Abstract

We present a straightforward introduction to the unconditional integrability in the extended sense. We state and proof useful necessary and sufficient conditions for unconditional integrability of complex or real valued functions. As an application, we obtain a simpler direct proof of mildly extended version of the Birkhoff's pointwise ergodic theorem.

Keywords: Conditional integrability; conditional expectation; ergodic theorem; pointwise convergence.

2010 mathematics subject classification: 38A05, 28D05, 28A25.

1 Introduction

The notion of integration is undeniably of fundamental importance in analysis and its application. The Lebesgue integral has been dominating the theory of integration since its introduction by Henri Lebesgue at the beginning of the 20^{th} century. New more efficient approach to integration has recently been introduced, see for example [1-3] replacing many of the cumbersome manipulations of the Lebesgue integral with simpler and concise arguments. This is due mainly by the non-dependency of the new integration theory on

**Corresponding author: E-mail: robdera@yahoo.com;*

the measure theory. Such an approach has led to various extensions and easy proofs of the classical results related to integration theory, see for example [4-7]. The main purpose of this paper is to give a thorough study of the concept of unconditional integrability in the setting of the new approach to integration. We obtain as an application an easy proof of a more general statement of the ergodic theorem.

Ergodic theory is the study of long term average behavior of systems evolving in time and related problems. Vaguely speaking, the ergodic theorem asserts that in an ergodic dynamical system, the statistical (or time) average is the same as the space average. The first result in this direction is the Poincaré recurrence theorem, which claims that almost all points in any subset of the phase space eventually revisit the set. Various ergodic theorems and their proofs have since been provided: [8-11]. Two of the most important ones are those of Birkhoff and von Neumann that were proved almost at the same time in early 1930's and which assert the existence of a time average along each trajectory. While von Neumann's result concerns L^2 -convergence, and has a quick proof, Birkhoff's theorem is about pointwise convergence, holds for any function in L^1 and has proofs that all require hard analysis. The result we are interested in is the later seemingly stronger ergodic theorem. We propose an ergodic theorem that is slightly more general than the Birkhoff's theorem. Our result is valid for any function that is not necessarily measurable but integrable in an extended sense and for any ergodic transformation that is not measurable either.

The paper is organized as follows. Section 2 will be entirely devoted to a thorough review of the notion of unconditional integrability. The concept of derivative of set functions as well as the notion of conditional expectation are reviewed in Section 3. In Section 4, we introduce and study ergodicity in a slightly extended setting and we show how the proof of the pointwise ergodic theorem is intrinsically related to the definition of unconditional integrability.

2 Unconditional Integrability

Throughout this paper, Ω will be an arbitrary nonempty set; 2^{Ω} will denote the power set of Ω, i.e. the set of all subsets of Ω, and $\mu : \Sigma \subset 2^{\Omega} \to \mathbb{K}$ (= $\mathbb{R}$ or $\mathbb{C}$) is an integrator, that is, a set function defined on a semiring Σ of subsets of Ω, that satisfies the following properties:

1. $\mu(\emptyset) = 0$;
2. $|\mu(A)| \leq |\mu(A \cup B)| \leq |\mu(A)| + |\mu(B)|$ for every pair $A, B \in \Sigma$.

We note that the above properties imply that an integrator is necessarily σ-subadditive, that is

$$\left|\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right)\right| \leq \sum_{n\in\mathbb{N}} |\mu(A_n)|$$

for every sequence $\{A_n\}$ in Σ.

In general, we require the set to be at least a semiring. The triplet (Ω, Σ, μ) will be called an ***integrator space***. For example,

- If Σ is a σ-algebra and μ is a measure on measurable space (Ω, Σ), then the measure space (Ω, Σ, μ) is an integrator space.
- If μ is an outer measure, then the triplet $(\Omega, 2^{\Omega}, \mu)$ is an integrator space.
- If Σ is the set of all bounded intervals in $\mathbb{R}$, and ℓ is the length function on Σ, then the triplet $(\mathbb{R}, \Sigma, \ell)$ is an integrator space.

The notion of a Σ, μ***-subpartition*** P of a set $A \in 2^{\Omega}$ is defined to be any finite collection of subsets of A elements of Σ satisfying

1. $|\mu(I)| < \infty$ for all $I \in P$;
2. $I \cap J = \emptyset$ whenever I and J are different sets in P.

We denote by $\cup P$ the subset of A obtained by taking the union of all elements of P. It is worth noticing that $\cup P$ is not necessarily equal to A. A Σ, μ-subpartition P is said to be ***tagged*** if for each $I \in P$, a point $t_I \in I$ is chosen. We say that P is ***unconditionally tagged*** if for each $I \in P$ the tagging point t_I is not necessarily an element of I but do belong to $\cup P$. We denote by $\Pi(\Omega, \Sigma, \mu)$ (respectively $\Pi_u(\Omega, \Sigma, \mu)$) the collection of all tagged (respectively unconditionally tagged) Σ, μ-subpartitions of the set A. Clearly, we have $\Pi(\Omega, \Sigma, \mu) \subset \Pi_u(\Omega, \Sigma, \mu)$.

The mesh or the norm of $P \in \Pi_u(\Omega, \Sigma, \mu)$ is defined to be

$$\|P\| = \max\{|\mu(I)| : I \in P\}.$$

If $P, Q \in \Pi_u(\Omega, \Sigma, \mu)$ we say that Q is a refinement of P and we write $Q \succ P$ if $\|Q\| \le \|P\|$ and $\cup Q \supset \cup P$. It is readily seen that such a relation does not depend on the tagging points. It is also easy to see that the relation is transitive on $\Pi_u(\Omega, \Sigma, \mu)$. If $P, Q \in \Pi_u(\Omega, \Sigma, \mu)$, we denote

$$P \vee Q := \{ I \setminus \cup Q, J \setminus \cup P, I \cap J : I \in P, J \in Q\}$$

Clearly, $P \vee Q \in \Pi_u(\Omega, \Sigma, \mu)$, $P \vee Q \succ P$ and $P \vee Q \succ Q$. Thus, the relation has the upper bound property on $\Pi_u(\Omega, \Sigma, \mu)$. We then infer that the set $\Pi_u(\Omega, \Sigma, \mu)$ is directed in the sense of Moore-Smith (as described by McShane [12]) by the binary relation $\succ$.

Given a function $f : \Omega \to K$, we associate the mapping $f_\mu : \Pi_u(A, \Sigma, \mu) \to K$ defined by

$$f_\mu(P) = \sum_{I \in P} f(t_I)\mu(I)$$

- If $P \in \Pi(A, \Sigma, \mu)$, the sum $f_\mu(P) = \sum_{I \in P} f(t_I)\mu(I)$ is called the ***Σ, μ-Riemann sum*** of f at P.
- If $P \in \Pi_u(A, \Sigma, \mu)$, the sum $f_\mu(P) = \sum_{I \in P} f(t_I)\mu(I)$ is called the ***unconditional Σ, μ-Riemann sum*** of f at P.

Since $\Pi_u(A, \Sigma, \mu)$ and $\Pi(A, \Sigma, \mu)$, are both directed by refinement $\succ$, the function f_μ is a net. We are now ready to give our definition of integrability.

Definition 2.1. A function $f : \Omega \to K$ is said to be ***Σ, μ-integrable,*** respectively ***unconditionally Σ, μ-integrable***, over a subset A of Ω if $\lim_{(\Pi(A,\Sigma,\mu),\succ)} f_\mu := \int_A f d\mu$, respectively $\lim_{(\Pi_u(A,\Sigma,\mu),\succ)} f_\mu := \int_A f d\mu$) exists in $\mathbb{K}$.

In other words, if for every $\epsilon > 0$, there exists $P_0 \in \Pi(A, \Sigma, \mu)$, respectively $P_0 \in \Pi_u(A, \Sigma, \mu)$, such that for every $P \in \Pi(A, \Sigma, \mu)$, respectively $P \in \Pi_u(A, \Sigma, \mu)$, $P \succ P_0$, we have

$$\left| f_\mu(P) - \int_A f d\mu \right| < \epsilon.$$

We denote by $I(A, \Sigma, \mu)$, respectively $I_u(A, \Sigma, \mu)$, the set of all functions $f : \Omega \to \mathbb{K}$ that are Σ, μ-integrable, respectively unconditionally Σ, μ-integrable, over a given subset A of Ω. We infer that being defined as limit operators, the two types of integral are both linear, and therefore the spaces $I(A, \Sigma, \mu)$ and $I_u(A, \Sigma, \mu)$ are both linear spaces. It is also clear that $I_u(A, \Sigma, \mu) \subset I(A, \Sigma, \mu)$. As expected, we shall see later that the inverse inclusion is not necessarily true.

We say that a function f is μ-***essentially equal*** on A to a function g and we write $f \sim g$ if $\mu(\{x \in A : f(x) \neq g(x)\}) = 0$. It is readily seen that the binary relation $\sim$ is an equivalence relation either of $\mathcal{I}(A, \Sigma, \mu)$ or $\mathcal{I}_u(A, \Sigma, \mu)$. The quotient spaces $\mathcal{I}(A, \Sigma, \mu)/\sim$ and $\mathcal{I}_u(A, \Sigma, \mu)/\sim$ shall be respectively denoted by $I(A, \Sigma, \mu)$ and $I_u(A, \Sigma, \mu)$. It is worth noticing that if A is a topological space, if Σ is a σ-algebra containing the Borel sets of A, and if μ is the Lebesgue measure on A, then $I_u(A, \Sigma, \mu) \cong L^1(A, \Sigma, \mu)$, the space of Lebesgue integrable functions over A.

For every $f: \Omega \to \mathbb{K}$, we define

- the Σ, μ-***variation*** of f over the set A to be

  $\mathrm{var}_{\Sigma,\mu}(f, A) := \sup\{|f_\mu(P)| : P \in \Pi(A, \Sigma, \mu)\}$;

- the ***unconditional*** Σ, μ-***variation*** of f over the set A to be

  $\mathrm{uvar}_{\Sigma,\mu}(f, A) := \sup\{|f_\mu(P)| : P \in \Pi_u(A, \Sigma, \mu)\}$.

We say that the function f is

- of ***bounded*** $\boldsymbol{\Sigma, \mu}$-***variation*** over A if $\mathrm{var}_{\Sigma,\mu}(f, A) < \infty$
- of ***bounded unconditional*** Σ, μ-***variation*** if $\mathrm{uvar}_{\Sigma,\mu}(f, A) < \infty$.

Clearly, if $f \in \mathcal{I}(A, \Sigma, \mu)$ then f is of bounded Σ, μ-variation and if $f \in \mathcal{I}_u(A, \Sigma, \mu)$ then f is of bounded unconditional Σ, μ-variation. We then define

$\|f\| = \mathrm{var}_{\Sigma,\mu}(f, A)$ and $\|f\|_u = \mathrm{uvar}_{\Sigma,\mu}(f, A)$,

It is readily seen that each of $f \mapsto \|f\|$ and $f \mapsto \|f\|_u$ defines a seminorm respectively on the space $\mathcal{I}(A, \Sigma, \mu)$ and $\mathcal{I}_u(A, \Sigma, \mu)$ and yields a norm respectively on $I(A, \Sigma, \mu)$ and $I_u(A, \Sigma, \mu)$. Moreover, the space $I(A, \Sigma, \mu)$ and $I_u(A, \Sigma, \mu)$ are Banach spaces.

We have the following proposition.

Proposition 2.1. *If $f: \Omega \to \mathbb{K}$ is Σ, μ-integrable, respectively unconditionally Σ, μ-integrable, over a given subset A of Ω, then for every $\epsilon > 0$, there exists $P_0 \in \Pi(A, \Sigma, \mu)$, respectively $P_0 \in \Pi_u(A, \Sigma, \mu)$, such that $|f_\mu(Q)| \leq \epsilon$ for every $Q \in \Pi(A, \Sigma, \mu)$, respectively $Q \in \Pi_u(A, \Sigma, \mu)$, that does not intersect P_0 and such that $\|Q\| \leq \|P_0\|$.*

Proof. Fix $\epsilon > 0$. Let $P_0 \in \Pi(A, \Sigma, \mu)$ such that we have

$$\left| f_\mu(P) - \int_A f d\mu \right| < \epsilon/2.$$

for $P_0 \in \Pi(A, \Sigma, \mu)$ such that $P \succ P_0$. Fix such a P. Then for every $Q \in \Pi(A, \Sigma, \mu)$ that does not intersect P_0 and such that $\|Q\| \leq \|P_0\|$, we have $P_0 \vee Q \in \Pi(A, \Sigma, \mu)$, $P_0 \vee Q \succ P_0$ and therefore

$$\left| f_\mu(P_0 \vee Q) - \int_A f d\mu \right| < \epsilon/2.$$

It follows that

$$\begin{aligned}\left|f_\mu(Q)\right| &= \left|f_\mu(P_0 \vee Q) - f_\mu(P_0)\right| \\ &\leq \left|f_\mu(P) - \int_A f d\mu\right| + \left|\int_A f d\mu - f_\mu(P_0)\right| < \epsilon.\end{aligned}$$

The unconditional case is proved in a similar fashion by simply using $\Pi_u(A,\Sigma,\mu)$ in lieu of $\Pi(A,\Sigma,\mu)$. The proof is complete. □

For arbitrary Σ,μ-subpartition P and Q, we shall denote

$$P \wedge Q := \{I \cap J : I \in P, J \in Q\} \quad \text{and} \quad P \triangle Q := P \vee Q \setminus P \wedge Q.$$

Definition 2.2. A function $f: \Omega \to \mathbb{K}$ is said to satisfy the ***Cauchy criterion for Σ,μ-integrability*** respectively the ***Cauchy criterion for unconditional Σ,μ-integrability,*** over a set $A \subset \Omega$ if for every $\epsilon > 0$, there exists $P_0 \in \Pi(A,\Sigma,\mu)$, respectively $P_0 \in \Pi_u(A,\Sigma,\mu)$, such that

$$\left|f_\mu(P \vee Q) - f_\mu(P \wedge Q)\right| \leq \epsilon$$

whenever P, Q are elements of $\Pi(A,\Sigma,\mu)$, respectively $\Pi_u(A,\Sigma,\mu)$, such that $P, Q \succ P_0$,.

We notice that if $P, Q \in \Pi(A,\Sigma,\mu)$ (respectively $\Pi_u(A,\Sigma,\mu)$), then

$$\left|f_\mu(P \vee Q) - f_\mu(P \wedge Q) = |f_\mu(P \wedge Q)|.\right|$$

The following fact is then easily derived.

Proposition 2.2. *A function $f: \Omega \to \mathbb{K}$ satisfies the Cauchy criterion for Σ,μ-integrability, respectively the Cauchy criterion for unconditional Σ,μ-integrability, over a set $A \subset \Omega$ if and only if for every $\epsilon > 0$, there exists $P_0 \in \Pi(A,\Sigma,\mu)$, respectively $\Pi_u(A,\Sigma,\mu)$, such that $\left|f_\mu(Q)\right| \leq \epsilon$ for every $Q \in \Pi(A,\Sigma,\mu)$, respectively $Q \in \Pi_u(A,\Sigma,\mu)$, that does not intersect P_0.*

We have seen (Proposition 2.1) that every Σ,μ-integrable, respectively unconditionally Σ,μ-integrable, function satisfies the Cauchy criterion for Σ,μ-integrability, respectively the Cauchy criterion for unconditional Σ,μ-integrability. It turns out that the converse also holds. This follows from the general well-known fact that for nets taking values in a Banach space, the Cauchy net condition is equivalent to the net convergence (see for example [12]). Clearly, the two Cauchy conditions introduced in Definition 2.2 correspond exactly to the Cauchy condition for the nets $\Pi(A,\Sigma,\mu) \ni P \mapsto f_\mu(Q) \in \mathbb{K}$ and $\Pi_u(A,\Sigma,\mu) \ni P \mapsto f_\mu(Q) \in \mathbb{K}$. It follows that

Proposition 2.3. *A function $f: \Omega \to \mathbb{K}$ satisfies*

- *the Cauchy criterion for Σ,μ-integrability over a set $A \subset \Omega$ if, and only if, $f \in I(A,\Sigma,\mu)$.*
- *the Cauchy criterion for unconditional Σ,μ-integrability over a set $A \subset \Omega$ if, and only if, $f \in I_u(A,\Sigma,\mu)$.*

The following result gives useful characterizations of unconditional integrability.

Theorem 2.1. *Let $f: \Omega \to \mathbb{K}$. The following statements are equivalent for a subset A of Ω:*

1. *f is unconditionally Σ, μ-integrable over A.*
2. *For any injection $\varpi : \Gamma \to A$, the function $\gamma \mapsto f(\varpi(\gamma))$ is Σ, η-integrable over Γ, where the integrator $\eta : \varpi^{-1}(\Sigma) \to \mathbb{K}$ is defined by $\eta(\varpi^{-1}(E)) = \mu(E)$ for all $E \in \Sigma$.*
3. *For every function $\sigma : A \to \{-1,1\}$, the function $t \mapsto \sigma(t)f(t)$ is Σ, μ-integrable over A.*
4. *For every bounded function $\varphi : A \to \mathbb{K}$, the function $t \mapsto \varphi(t)f(t)$ is Σ, μ-integrable over A.*

Proof. We have 4. ⇒ 3. and 2 . ⇒ 1. are obvious. To see 1. ⇒ 2., suppose $f \in I_u(A, \Sigma, \mu)$ and let $\epsilon > 0$. Then there exists $P_0 \in \Pi_u(A, \Sigma, \mu)$ such that $|f_\mu(R)| < \epsilon$ for every $R \in \Pi_u(A, \Sigma, \mu)$ that does not intersect P_0. Let $\varpi : \Gamma \to A$ be an injective mapping. We can choose $Q_0 \in \Pi_u(\Gamma, \varpi^{-1}(\Sigma), \eta)$ so that $(Q_0) \succ P_0$. Again, by injectivity of ϖ, if $Q \in \Pi_u(\Gamma, \varpi^{-1}(\Sigma), \eta)$ and $Q \cap Q_0 = \emptyset$ then $\varpi(Q) \cap \varpi(Q_0) = \emptyset$ and therefore $\varpi(Q)$ is disjoint from $P_{0,}$ and we have

$$|f \circ \varpi_\mu(Q)| = |f_\mu \varpi(Q)| < \epsilon$$

Hence, the function $\gamma \mapsto f(\varpi(\gamma))$ is $\varpi^{-1}(\Sigma)$,η-integrable over Γ. We have established that 1. ⇒ 2..

To show that 2. ⇔ 3., let $\Gamma_1 = \sigma^{-1}(1)$ and $\Gamma_{-1} = \sigma^{-1}(-1)$. Then $A = \Gamma_1 \cup \Gamma_{-1}$ and $\Gamma_1 \cup \Gamma_{-1} = \emptyset$. Let $\varpi_1 : \Gamma_1 \to A$ and $\varpi_{-1} : \Gamma_{-1} \to A$ be respectively, the canonical injection of Γ_1 and Γ_{-1} into *A*. Then the following two functions

$$\gamma \in \Gamma_1 \mapsto f(\varpi_1(\gamma)) = f(\gamma) \text{ and } \gamma \in \Gamma_{-1} \mapsto f(\varpi_{-1}(\gamma)) = f(\gamma)$$

are both Σ, μ-integrable over *A* if and only if

$$\gamma \mapsto \sigma(t)f(\gamma) = 1_{\Gamma_1}(\gamma)f(\gamma) + 1_{\Gamma_{-1}}(\gamma)f(\gamma)$$

is Σ, μ-integrable over *A*.

3. ⇒ 4. We give the proof for the real case. The changes for complex case are straightforward. Let $\varphi : A \to \mathbb{K}$ be bounded and fix $P \in \Pi(A, \Sigma, \mu)$. Let $\sigma : A \to \{-1,1\}$ be defined by $\sigma(t) = \operatorname{sgn}(f(t)\mu(I))$. Then

$$\begin{aligned} \left|\sum_{I \in P} \phi(t_I) f(t_I)\, \mu(I)\right| &\le \sum_{I \in P} |\phi(t_I)||f(t_I)\mu(I)| \le \sup_{t \in A}|\phi(t)| \sum_{I \in P} |f(t_I)\mu(I)| \\ &\le \sup_{t \in A}|\phi(t)| \sum_{I \in P} \sigma(t_I) f(t_I)\mu(I) \le \sup_{t \in A}|\phi(t)| \left|\sum_{I \in P} \sigma(t_I) f(t_I)\mu(I)\right| \end{aligned}$$

The desired result follows. The proof is complete. □

Corollary 2.1. *A function $f : \Omega \to \mathbb{K}$ is unconditionally Σ, μ-integrable over A if and only if it is unconditionally Σ, μ-integrable over all subsets of A.*

Theorem 2.2. *If $f : \Omega \to \mathbb{K}$ is unconditionally Σ, μ-integrable over A, then for any injection $\varpi : \Gamma \to A$*

$$\int_A f d\mu = \int_{\varpi^{-1}(A)} f \circ \varpi d\eta$$

where the integrator $\eta : \varpi^{-1}(\Sigma) \to \mathbb{K}$ is defined by $\eta(\varpi^{-1}(E)) = \mu(E)$ for all $E \in \Sigma$.

Proof. Assume that $\epsilon > 0$. Choose $Q_1 \in \Pi(\varpi^{-1}(A), \varpi^{-1}(\Sigma), \eta)$ such that

$$\left|\int_{\varpi^{-1}(A)} f\circ\varpi d\eta - f\circ\varpi_{\eta}\big(Q_1\big)\right| < \frac{\epsilon}{3}.$$

Choose $P_1 \in \Pi_u(A,\Sigma,\mu)$ such that $P_1 \succ \varpi(Q_1)$ and

$$\left|f_{\mu}(P_1) - \int_A f d\mu\right| < \frac{\epsilon}{3}.$$

By injectivity, we can choose $Q_2 \in \Pi(\varpi^{-1}(A),\varpi^{-1}(\Sigma),\eta)$ such that $\varpi(Q_2) \succ P_1$ and

$$\left|\int_{\varpi^{-1}(A)} f\circ\varpi d\eta - f\circ\varpi_{\eta}\big(Q_2\big)\right| < \frac{\epsilon}{3}.$$

Choose $P_2 \in \Pi_u(A,\Sigma,\mu)$ such that $P_2 \succ \varpi(Q_2)$ and

$$\left|f_{\mu}(P_1) - \int_A f d\mu\right| < \frac{\epsilon}{3}.$$

Continuing in this way, we construct sequences $n \mapsto P_n$ and $n \mapsto Q_n$ such that $\varpi(Q_{n+1}) \succ P_n \succ \varpi(Q_n)$

$$\left|\int_{\varpi^{-1}(A)} f\circ\varpi d\eta - f\circ\varpi_{\eta}\big(Q_n\big)\right| < \frac{\epsilon}{3} \quad\text{and}\quad \left|f_{\mu}(P_n) - \int_A f d\mu\right| < \frac{\epsilon}{3}.$$

Now we let $H = \bigcup_{n\in\mathbb{N}} Q_n$ and define $\varpi' : H \to A$ by $\varpi'(t) = \varpi(t)$.

By our hypothesis, the function $t \mapsto f\big(\varpi'(t)\big)$ is also Σ,μ-integrable. On the other hand, it follows from the above two inequalities that

$$\left|\int_{\varpi^{-1}(A)} f\circ\varpi d\eta - f\circ\varpi'_{\eta}\big(Q_n\big)\right| < \frac{\epsilon}{3} \quad\text{and}\quad \left|f_{\mu}\big(\varpi'(Q_n)\big) - \int_A f d\mu\right| < \frac{\epsilon}{3}.$$

We notice that

$$f\circ\varpi'_{\eta}(Q_n) = \sum_{I\in Q_n} f(\varpi'(t_I))\eta(t_I) = \sum_{\varpi'(I)\in\varpi'(Q_n)} f(\varpi'(t_I))\mu(\varpi'(I)) = f_{\mu}(Q_n).$$

By the uniqueness of limit, we must have $\int_A f d\mu = \int_{\varpi^{-1}(A)} f\circ\varpi d\eta$ as to be shown. □

It follows that if $f \in \mathcal{I}_u(A,\Sigma,\mu)$, then in particular, both f and $|f|$ are in $\mathcal{I}(A,\Sigma,\mu)$. We shall denote by $\mathcal{I}^1(A,\Sigma,\mu)$ the space of all functions such that both f and $|f|$ are in $\mathcal{I}(A,\Sigma,\mu)$. It is then easy to see that $\mathcal{I}_u(A,\Sigma,\mu) = \mathcal{I}^1(A,\Sigma,\mu)$. More generally, for $1 \le p < \infty$, we denote by $\mathcal{I}^p(A,\Sigma,\mu)$, respectively by $I^p(A,\Sigma,\mu)$, the space of all functions (all classes of functions) such that $|f|^p$ are in $\mathcal{I}(A,\Sigma,\mu)$, respectively in $I(A,\Sigma,\mu)$. By a standard argument, one shows that $\mathcal{I}^p(A,\Sigma,\mu)$ and $I^p(A,\Sigma,\mu)$ are complete spaces.

For the particular case where Σ is a σ-algebra containing the Borel sets, μ is the Lebesgue measure on Σ, and A a measurable subset of Ω, then the space of unconditionally Σ,μ-integrable $I^1(A,\Sigma,\mu)$, corresponds exactly to the Lebesgue function space $L^1(A,\Sigma,\mu)$.

We finish this section by noticing that if $\Sigma_1 \subset \Sigma_2$ in 2^Ω, then $\Pi(A, \Sigma_1, \mu) \subset \Pi(A, \Sigma_2, \mu)$ (respectively, $\Pi_u(A, \Sigma_1, \mu) \subset \Pi_u(A, \Sigma_2, \mu)$. Hence, we have the following proposition stating the relationship between Σ_1,μ-integrability and Σ_2,μ-integrability. (See [2] for more details.)

Proposition 2.4. *Assume that $\Sigma_1 \subset \Sigma_2$ in 2^Ω. Then for every $A \in 2^\Omega$, $\mathcal{I}(A, \Sigma_1, \mu) \subset \mathcal{I}(A, \Sigma_2, \mu)$, $\mathcal{I}_u(A, \Sigma_1, \mu) \subset \mathcal{I}_u(A, \Sigma_2, \mu)$) and for $f \in \mathcal{I}(A, \Sigma_1, \mu)$*

$$\int_A f d\mu_{\Sigma_1} = \int_A f d\mu_{\Sigma_2}.$$

3 Derivative of Set Functions and Conditional Excpectations

In this section, we discuss the notion of derivatives of set functions and conditional expectation. Let (Ω, Σ, μ) be a non-negative finite integrator space and let $\mathcal{F}$ be a sub-σ-algebra of the σ-algebra Σ. Let $\mu_{\mathcal{F}}$ denote the restriction of μ to $\mathcal{F}$. Then $(\Omega, \mathcal{F}, \mu_{\mathcal{F}})$ is a non-negative finite integrator space. We infer from Proposition 2.4 that $\mathcal{I}(\Omega, \mathcal{F}, \mu_{\mathcal{F}}) \subset \mathcal{I}(\Omega, \Sigma, \mu)$. In what follows, we shall simply write $\mathcal{I}(\Omega, \mathcal{F}, \mu)$ in lieu of μ and $\int f d\mu$ in lieu of $\int f d\mu_{\mathcal{F}}$.

Definition 3.1. Let μ be a non-negative integrator on (Ω, Σ). We say that a set function $\nu: \Sigma \to \mathbb{K}$ is ***absolutely continuous*** with respect to μ and we write $\nu \ll \mu$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $|\nu(E)| < \epsilon$ whenever $E \in \Sigma$ with $\mu(E) < \delta$.

For example, if $f \in \mathcal{I}(A, \Sigma, \mu)$ then the set function ν defined by

$$\nu(E) = \int_E f d\mu$$

is absolutely continuous with respect to μ. The following theorem, which is a particular case of Theorem 10 of [4] states that essentially, all μ-absolutely continuous integrators occur in this way.

Theorem 3.1. *Let (Ω, Σ, μ) be a non-negative integrator space. Let ν be an integrator on (Ω, Σ) with the property $\nu \ll \mu$. Then there exists a function $f \in \mathcal{I}(A, \Sigma, \mu)$ such that*

$$\nu(E) = \int_E f d\mu$$

f*or all $E \in \Sigma$. Moreover, f is unique in the sense that if g is a function with the same property then $f = g$ μ-almost everywhere.*

It is worth noticing that unlike the Radon-Nikodým derivative of a measure, the above density function f does not need to be Lebesgue integrable, let alone measurable. Such a function f will simply be called the μ-***derivative*** of the set function ν. In particular, if $\mathcal{F} \subset \Sigma$ is a sub-σ-algebra and $f \in \mathcal{I}(A, \Sigma, \mu)$ (not necessarily absolutely integrable) then the set function $\nu: \Sigma \to \mathbb{K}$ defined by $\nu(E) = \int_E f d\mu$ is μ-absolutely continuous. Its μ-derivative is denoted by $E(f|\mathcal{F})$ and is called ***conditional expectation of*** f with respect to $\mathcal{F}$.

We thus have the following result:

Theorem 3.2. *Let (Ω, Σ, μ) be a non-negative finite integrator space and $\mathcal{F} \subset \Sigma$* is a *sub-$\sigma$-algebra. Any function $f \in \mathcal{I}(A, \Sigma, \mu)$ admits a unique conditional expectation $E(f|\mathcal{F})$.*

Again, being a μ-derivative, the conditional expectation is not required to be measurable. Nonetheless, it does have most of the basic properties of the classical conditional expectation for Lebesgue integrable functions. Namely,

1. The mapping $f \mapsto E(f|\mathcal{F})$ is linear.
2. If g is $\mathcal{F}$ -measurable and $|g| < \infty$ μ-almost everywhere, then $E(gf|\mathcal{F}) = gE(f|\mathcal{F})$.
3. $E(f|\Sigma) = f$.
4. $\int_E E(f|\mathcal{F})d\mu = \int_E fd\mu$ for every $E \in \mathcal{F}$.
5. If $\mathcal{N}$ denotes the σ-algebra consisting of all subsets B in Σ such that $\mu(B) = 0$ or $\mu(B) = \mu(\Omega)$, then $E(f|\mathcal{N}) = \int_\Omega fd\mu$.

4 Extended Ergodic Theorem

In this section, we introduce a mild extension of the concept of ergodicity. We agree to say that an integrator space (Ω, Σ, μ) is a ***state system*** if the set system Σ contains Ω and the integrator μ is a non-negative finite integrator, that is, $\mu: \Sigma \to [0, +\infty)$.

If (Ω, Σ, μ) is a state system, then the set function $\mu^*: 2^\Omega \to [0, +\infty)$ given by

$$\mu^*(B) = \inf\left\{\sum_{I\in P} \mu(P): P \in \Pi(\Omega, \Sigma, \mu)\right\}$$

is a well-defined integrator that clearly seen to extend the integrator μ to the whole power set 2^Ω. It will be called the ***size-function*** associated to μ. It follows from Proposition 2.4 that if $f \in \mathcal{I}(\Omega, \Sigma, \mu)$ then $f \in \mathcal{I}(2^\Omega, \Sigma, \mu^*)$ and one has

$$\int_B fd\mu = \int_B fd\mu^*.$$

or all $B \in 2^\Omega$.

The following definition slightly extends the concept of *measure-preserving transformation*.

Definition 4.1. Let $(\Omega, \Sigma_i, \mu_i)$ where i = 1,2 be two state systems. A map $T: \Omega_1 \to \Omega_2$ is said to be ***size-preserving*** if ${\mu_1}^*(T^{-1}B) = \mu_2(B)$ for all $B \in \Sigma_2$.

In particular, a map $T: \Omega \to \Omega$ is size-preserving if $\mu^*(T^{-1}B) = \mu(B)$ for all $B \in \Sigma$. Observe that our definition does not require the measurability of the transformation T.

Our next definition is a mild extension of the concept of ergodicity.

Definition 4.2. Let (Ω, Σ, μ) be a state system. Then a size-preserving map $T: \Omega \to \Omega$ is said to be ***ergodic*** if for every $B \in \Sigma$, whenever $T^{-1}B = B$, then $\mu(B) = 0$ or $\mu(B) = \mu(\Omega)$.

Again, the measurability of the transformation T is not required in the above definition of ergodicity. The following lemma is useful.

Lemma 4.1. *Let (Ω, Σ, μ) be a state system. Let $T: \Omega \to \Omega$ be ergodic. Then for every $B \in \Sigma$ such that $\mu^*(T^{-1}B \vartriangle B) = 0$, one has $\mu(B) = 0$ or $\mu(B) = \mu(\Omega)$.*

Proof. For each $k \in \mathbb{N}$, we have

$$T^{-1}B \vartriangle B \subset \bigcup_{i=0}^{k-1} T^{-(i+1)}B \vartriangle T^{-i}B = \bigcup_{i=0}^{k-1} T^{-i}\left(T^{-1}B \vartriangle B\right).$$

Since T preserves sizes, we have

$$\mu^*(T^{-1}B \vartriangle B) \leq \sum_{i=0}^{k-1} \mu^*\left(T^{-i}(T^{-1}B \vartriangle B)\right) = k\,\mu^*(T^{-1}B \vartriangle B) = 0.$$

Let $C = \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} T^{-j}B$. Then clearly, $T^{-1}C = C$ and since T is ergodic, we have $\mu(C) = 0$ or $\mu(\Omega)$. We also have

$$\mu^*\left(B \vartriangle \bigcup_{j=n}^{\infty} T^{-j}B\right) \leq \sum_{j=n}^{\infty} \mu^*(T^{-1}B \vartriangle B) = 0.$$

Since $B \vartriangle C \subset B \vartriangle \bigcup_{j=n}^{\infty} T^{-j}B$, we have $\mu^*(B \vartriangle C) = 0$. In particular, we have $\mu^*(B \setminus C) = \mu^*(C \setminus B) = 0$, and

$$\mu^*(C \cap B) \leq \mu^*(C) \leq \mu^*(C \setminus B) + \mu^*(C \cap B) = \mu^*(C \cap B).$$

Hence $\mu^*(C) = \mu^*(C \cap B)$. By a symmetric argument, we also obtain $\mu^*(B) = \mu^*(C \cap B)$ and therefore we have $\mu^*(C) = \mu^*(B) = 0$ or $\mu(\Omega)$. □

We denote by $\mathcal{F}(\Omega)$ the set of all complex valued functions on non-empty set Ω. Let $(\Omega_i, \Sigma_i, \mu_i)$ where $i = 1,2$ be two state systems, and let $T: \Omega_1 \to \Omega_2$ be a size-preserving transformation. Define an operator $U_T: \mathcal{F}(\Omega_1) \to \mathcal{F}(\Omega_2)$ by $U_T(f) = f \circ T$. The following facts about the operator U_T are easily checked.

1. U_T is a linear operator.
2. $U_T c = c$ where c is a constant.
3. If $f \in \mathcal{I}(\Omega_2, \Sigma_2, \mu_2)$, then $U_T(f) \in \mathcal{I}(\Omega_1, \Sigma_1, \mu_1)$ and $\int_{\Omega_1} U_T(f) d\mu_1 = \int_{\Omega_2} f d\mu_2$.
4. Let $p \geq 1$. Then $U_T \mathcal{I}^p(\Omega_2, \Sigma_2, \mu_2) \subset \mathcal{I}^p(\Omega_1, \Sigma_1, \mu_1)$.

Using the above properties, one obtains the following characterizations of ergodicity.

Theorem 4.1. *Let T be a size preserving transformation on a state system (Ω, Σ, μ). The following statements are equivalent:*

1. *T is ergodic.*
2. *If $f \in \mathcal{F}(\Omega)$ satisfies $U_T f(x) = f(x)$ for μ^*-almost every $x \in \Omega$, then f is μ^*-almost everywhere equal to a constant function.*
3. *If $f \in \mathcal{I}^p(\Omega, \Sigma, \mu)$ satisfies $U_T f(x) = f(x)$ for μ^*-almost every $x \in \Omega$, then f is μ^*-almost everywhere equal to a constant function.*

Proof. 1. ⇒ 2. Suppose $f\,(T(x)) = f(x)$ for μ^*-almost every $x \in \Omega$. Assume without loss of generality that f is real valued (otherwise, we consider separately the real part and the imaginary part of f). For each $n \geq 1$ and $k \in \mathbb{Z}$, define

$$B_{n,k} = \left\{x \in \Omega: \frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n}\right\}$$

Since T is ergodic, Lemma 4.1 implies that $\mu^*(B_{n,k}) = 0$ or $\mu(\Omega)$. In fact, being non-decreasing and bounded above, the sequence $n \mapsto \frac{k}{2^n}$ converges. Since for each $n \geq 1$, one has $\Omega = \bigcup_{k\in\mathbb{Z}} B_{n,k}$, there exist $k_n \in \mathbb{Z}$ such that $\mu^*(B_{n,k_n}) = \mu^*(\Omega)$. Let $B = \bigcap_{n\in\mathbb{N}} B_{n,k_n}$. Then $\mu^*(B) = \mu^*(\Omega)$, and if $x \in B$, then $\big|f(x) -$ $k2n<12n$ for all $n\in N$. Hence $fx=\lim n\to\infty k2n$ and f is $\mu*$-almost everywhere equal to a constant function.

2. $\Rightarrow$ 3. is obvious. For 3. $\Rightarrow$ 1., suppose $T^{-1}B = B$ and $\mu(B) > 0$. Then the indicator function $1_B \in \mathcal{I}^p(\Omega, \Sigma, \mu)$ and $1_B \circ T = 1_{T^{-1}B} = 1_B$. Hence, by 3., 1_B is μ^*-almost everywhere equal to constant function, i.e. $1_B = 1_\Omega$ μ-almost everywhere and therefore $\mu(B) = \mu(\Omega)$. The proof is complete. □

As a way of application, we now state and prove our promised extended version of the Birkhoff's Ergodic Theorem.

Theorem 4.2. *Let T be a size-preserving transformation on a state system (Ω, Σ, μ). We have*

1. *If $f \in \mathcal{I}^1(\Omega, \Sigma, \mu)$, then* $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(T^i x)$ *exists for* μ-almost every $x \in \Omega$.
2. *If $f \in \mathcal{I}^1(\Omega, \Sigma, \mu)$ and T is ergodic, then* $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(T^i x) = \frac{1}{\mu(\Omega)}\int_\Omega f d\mu$.

Proof. Let $f \in \mathcal{I}^1(\Omega, \Sigma, \mu)$ and let $B = \{x \in \Omega : U_T f(x) \neq f(x)\}$.

- If $\mu^*(\Omega \setminus B) = 0$, then f is μ-almost everywhere constant and therefore for μ-almost everywhere x, the sum $\frac{1}{n}\sum_{i=0}^{n-1} f(T^i x)$ is constant and so $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(T^i x)$ exists for μ-almost every $x \in \Omega$.
- If $\mu^*(B) = 0$, then for n large enough, the expression

$$\frac{\mu^*(\Omega)}{n}\sum_{i=0}^{n-1} f\left(T^i x\right) = \sum_{i=0}^{n-1} f\left(T^i x\right)\frac{\mu^*(\Omega)}{n}$$

is an unconditional 2^Ω, $\mu*$-Riemann sum of f associated to a 2^Ω, $\mu*$-subpartition P of Ω consisting of subsets of size equal to $\frac{\mu^*(\Omega)}{n}$. The tagging points being chosen to be the elements $x, Tx, T^2x, \dots, T^{n-1}x$ of the orbit of x. Since $f \in \mathcal{I}^1(\Omega, \Sigma, \mu)$ then $f \in \mathcal{I}^1(\Omega, 2^\Omega, \mu)$ therefore $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f\left(T^i x\right) = \frac{1}{\mu(\Omega)}\int_\Omega f d\mu$ exists. This completes the proof of part 1.

For part 2., we need to consider the σ-algebra of Σ_T of T-invariant subsets, namely

$$\Sigma_T = \left\{B \in T : T^{-1}B = B \ \mu \text{ almost everywhere}\right\}$$

If T is ergodic, then the σ-algebra Σ_T corresponds exactly to the the σ-algebra $\mathcal{N}$ consisting of all subsets B in Σ such that $\mu(B) = 0$ or $\mu(\Omega)$. Therefore

$$E(f|\Sigma_T) = E(f|\mathcal{N}) = \int_\Omega f d\mu.$$

On the other hand, since $f \in \mathcal{I}^1(\Omega, \Sigma, \mu)$, we have $\int_\Omega f d\mu = \int_\Omega f d\mu^*$. The result follows from part 1. □

5 Conclusion

This paper essentially gives useful characterizations of unconditional integrability of scalar valued functions with respect to a non-negative integrator. As an application, a simpler statement of a more general ergodic theorem is obtained. The author believes that the interest of this paper lies not only in the obtained results, but also in the light it shed on the very foundation of the study of integration theory.

Competing Interests

The authors declare that no competing interests exist.

References

[1] Robdera MA. Unified approach to vector valued integration. Int. J. Fun. Ana., Operator Theory & Appl. 2013a;5(2):119-139.

[2] Robdera MA. A new comprehensive approach to vector valued stochastic integration. Int. J. Modeling & Optimization. 2014a;4(4):299-304.
DOI: 10.7763/IJMO.2014.V4.389

[3] Robdera MA. Tensor integral: A new comprehensive approach to the integration theory. British J. Math. & Comp. Sci. 2014b;4(22):3236-3244.
DOI: 10.9734/BJMCS/2014/12274

[4] Robdera MA, Kagiso DN. On the differentiability of vector valued set function. Advanced in Pure Mathematics. 2013b;3:653-659.
DOI: 10.4236/apm.2013.38087

[5] Robdera MA. On Non-metric covering lemmas and extended lebesgue-differentiation theorem. British J. Math. & Comp. Sci. 2015;8(3):220-228.
DOI: 10.9734/BJMCS/2015/16752

[6] Robdera MA. Extension of the Lusin's theorem, the Severini-Egorov's theorem and the Riesz subsequence theorem. Asian J. Mathematics. 2016a;1(4):1-9.
DOI: 10.9734/ARJOM/2016/29547

[7] Robdera MA. On the Riesz representation theorem and integral operators. Quaestines Mathematicae. 2016b;39(4):445-455.
DOI: 10.2989/16073606.2015.1091045

[8] Birkhoff GD. Proof of the ergodic theorem, Proc. Nat. Acad. Sci. USA. 1931;17(12):656-660. PMCID: PMC1076138

[9] Von Neumann J. Proof of the quasi-ergodic hypothesis. Proc. Nat. Acad. Sci. USA 1932;18(1):7082

[10] Kamae Teturo, Keane, Michael. A simple proof of the ratio ergodic theorem. Osaka J. Math. 1997; 34(3):653-657.
Available: http://hdl.handle.net/11094/12364

[11] Moore CC. Ergodicity of flows on homogeneous spaces. Amer. J. Math. 1966;88(1):154-178.
DOI: 10.2307/2373052

[12] McShane EJ. Partial orderings and Moore-Smith Limits. Amer. Math. Monthly. 1952;59:1-11.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://sciencedomain.org/review-history/20405