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Combination of Ramadan Group and Reduced Differential Transforms for Partial Differential Equations with Variable Coefficients

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Authors' contributions

This work was carried out in collaboration between both authors. Author ARH designed the study, performed the analysis, wrote the protocol and wrote the first draft of the manuscript. Author AKM studied, wrote the manuscript and managed the analyses of the study. Both authors approved the final manuscript.

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Abstract

In this article, an analytical technique is provided to solve partial differential equations with variable coefficients. This technique is a combination of the integral transform known as Ramadan group integral transform with the reduced differential transform. The method can easily be applied to many nonlinear problems and is capable of reducing the size of computational work to overcome the deficiency that is caused by of the nonlinear terms that can not be handled using the known integral transforms alone. Illustrative examples are examined to support the proposed method. The results reveal that the suggested method is simple and effective.

Keywords: Ramadan Group Transform (RGT); reduced differential transform; partial differential equations with variable coefficients.

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1 Introduction

Different scientific and Physical problems have been modeled mathematically by systems of Ordinary Differential equations. Many authors have solved system of Ordinary Differential equations using different methods and techniques, see [1-4]. Among these physical problems which have received much attention are heat and wave equations. Wazwaz [5] applied the Adomian method for solving such problems with variable coefficients. The main analytical approach in literature is Adomian method [6] and Ramadan group transform method [7,8]. Momani and Qaralleh [9] applied the method to the time fractional heat-like and wave-like equations with variable coefficients. In the literature there are numerous integral transforms and widely used in physics, astronomy as well as in engineering. In order to solve the differential equations, the integral transform were extensively used and thus there are several works on the theory and application of integral transform such as the Laplace, Fourier, Mellin, and Hankel, to name but a few. In this paper, a combination of the new integral transform known as Ramadan group integral transform with the reduced differential transform is proposed to solve partial differential equations with variable coefficients. This paper is organized as follows. In section 2, definition of Ramadan group integral transform and its application to function derivatives is presented. Projected differential transform (PDTM) with its basic operations of PDTM is introduced in section 3. In section 4, the application of the incipient analytical technique to solve partial differential equations with variable coefficients is illustrated by solving several examples.

2 Ramadan Group Integral Transform (RGT) [7,8]

A new integral RG transform defined for functions of exponential order, is introduced. The proposed integral transform is a generalization of both Laplace and sumudu transforms. We consider functions in the set A, defined by:

$$A = \left\{ f(t) : \exists M, t_1, t_2 > 0 \ \ s.t. \left| f(t) \right| < M\, e^{\frac{|t|}{t_n}}, if \ \ t \in (-1)^n \times [0, \infty) \right\}.$$

The RG transform is defined by

$$K(s,u) = RG(f(t)) = \begin{cases} \int_0^\infty e^{-st} f(ut)\, dt, & 0 \le u < t_2, \\ \int_0^\infty e^{-st} f(ut)\, dt, & t_1 < u \le 0, \end{cases}$$

where s and u are complex variables with s and u are the transform variables for x and t, respectively.

This transform which is introduced by Raslan et al. [7] which is coupled with projected differential transform to solve nonlinear partial differential equations, see Ramadan and Hadhoud [8] is considered to be generalization to Laplace integral transform [10] and Sumudu integral transform introduced by Watugala [11].

Remarks:

1- If we set $u = 1$, we get the special case of Laplace transform

$$F(s) = L[f(t); s] = \int_0^\infty e^{-st} \ f(t) \ dt.$$

2- If we set $s = 1$, we get the special case of Sumudu transform

$$G(u) = S\left[f(t);u\right] = \int_0^\infty f(ut)e^{-t}dt, \qquad u \in \left(-\tau_1, \tau_2\right).$$

The Ramadan Group Integral Transform of function derivatives:

The following relations can be easily verified.

$$RG[\frac{df(t)}{dt}] = \frac{sRG[f(t)] - f(0)]}{u}.$$

$$RG[\frac{d^2 f(t)}{dt^2}] = \frac{s^2 RG[f(t)] - sf(0) - uf'(0)}{u^2}.$$

$$\vdots$$

$$RG[\frac{d^n f(t)}{dt^n}] = \frac{s^n RG[f(t)]}{u^n} - \sum_{k=0}^{n-1} \frac{s^{n-k-1} f^{(k)}(0)}{u^{n-k}}.$$

Table 1. Ramadan group, Laplace and Sumudu transforms of some functions

$f(t)$	$F(s) = L\{f(t)\}$	$G(u) = S\{f(t)\}$	$RG(f(t)) = K(s,u)$
1	$\frac{1}{s}$	1	$\frac{1}{s}$
t	$\frac{1}{s^2}$	u	$\frac{u}{s^2}$
$\frac{t^{n-1}}{(n-1)!}, n = 1, 2...$	$\frac{1}{s^n}$	u^{n-1}	$\frac{u^{n-1}}{s^n}$
$\frac{t^{a-1}}{\Gamma(a)}, \alpha > 0$	$\frac{1}{s^a}$	u^{a-1}	$\frac{u^{a-1}}{s^a}$

3 Projected Differential Transform Method (PDTM)

The definitions and operations of projected differential transform that can be avilabe in some papers see for example [12,13] is introduced as follows:

Definition 3.1

Assume the function $u(x,t)$ is both analytic and continuously differentiable with respect to time t and space x, then

$$U(x,k) = \frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial t^k}\right]_{t=0}, \tag{3.1}$$

is the transformed function of u(x ,t).

Definition 3.2

The projected differential inverse transform of $U(x,k)$ is defined as follows:

$$u(x,t)=\sum_{k=0}^{\infty}U(x,k)t^k \quad . \tag{3.2}$$

Then combining equation (3.1) and (3.2) we write

$$u(x,t)=\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}t^k \quad . \tag{3.3}$$

Table 2. Some operations of PDTM

Functional Form	Transformed Form
$u(x,t)$	$U(x,k)=\frac{1}{k!}[\frac{\partial^k}{\partial t^k}u(x,t)]_{t=0}$
$u(x,t)\pm v(x,t)$	$U(x,k)\pm V(x,k)$
$cu(x,t)$	$cU(x,k)$
$[\frac{\partial u(x,t)}{\partial x}]$	$[\frac{\partial}{\partial x}U(x,k)]$
$[\frac{\partial^r}{\partial x^r}u(x,t)]$	$\frac{(k+r)!}{k!}U(x,k+r)$
$u(x,t)v(x,t)$	$\sum_{r=0}^{k}U(x,r)V(x,k-r)$

4 Applications to the RGTM Coupled with PDTM

In order to show the effectiveness of the RGTM coupled with PDTM for solving the nonlinear partial differential equations, several examples are demonstrated. For all illustrative examples, we choose examples that have exact solutions.

Example 1

Consider the initial boundary value problem [5].

$$y_t=\frac{1}{2}x^2y_{xx} \ , \ 0<t<1 \ ,t>0, \tag{4.1}$$

with the boundary conditions

$$y(0,t)=0, \qquad y(1,t)=e^t, \tag{4.2}$$

and the initial condition

$$y(x,0)=x^2. \tag{4.3}$$

Taking Ramadan Group transform to both sides of equation (4.1) we have

$$\frac{sRG[y(x,t)]-y(x,0)}{u}=\frac{1}{2}x^2RG[y_{xx}(x,t)]$$

Using the initial condition (4.3), we get

$$sRG[y(x,t)]=x^2+\frac{u}{2}x^2RG[y_{xx}(x,t)]$$

That is,

$$RG[y(x,t)]=\frac{x^2}{s}+\frac{u}{2s}x^2RG[y_{xx}(x,t)] \tag{4.4}$$

Applying the inverse Ramadan Group transform of Eq. (4.4) implies that

$$y(x,t)=x^2+RG^{-1}\{\frac{u}{2s}x^2RG[y_{xx}(x,t)]\}$$

Using the reduced differential transform method, this leads to the recursive relation

$$y_{m+1}(x,t)=RG^{-1}\{\frac{u}{2s}x^2RG[A_m]\} \quad and \quad y_0(x,t)=x^2 \tag{4.5}$$

where $A_m=\dfrac{d^2}{dx^2}y(x,m)$ is the reduced differential transform of $y_{xx}(x,t)$.

For $m=0$ $, we\ get\ A_0=2 \quad and\ we\ then\ get\ y_1=RG^{-1}\{\frac{u}{2s}x^2RG[2]\}=x^2t$

For $m=1\ \ we\ \ get\ \ A_1=2t \quad and\ one\ gets \quad y_2=RG^{-1}\{\frac{u}{2s}x^2RG[2t]\}=x^2\frac{t^2}{2!}$

For $m=2 \quad we\ obtain\ A_2=t^2\ and\ hence\ \ y_3=RG^{-1}\{\frac{u}{2s}x^2RG[t^2]\}=x^2\frac{t^3}{3!}$

And so on, then the analytical solution of the problem (4.1 - 4.3) is given as:

$$y(x,t)=x^2t+x^2\frac{t^2}{2!}+x^2\frac{t^3}{3!}+\ldots=x^2e^t.$$

Example 2

Consider the two-dimensional heat-like model [5].

$$f_t = \frac{1}{2}(y^2 f_{xx} + x^2 f_{yy}), \quad 0 < x, y < 1,\ t > 0, \tag{4.6}$$

Subject to Neumann boundary conditions

$$f_x(0, y, t) = 0, \qquad f_x(1, y, t) = 2\sinh t,$$

$$f_y(x, 0, t) = 0, \qquad f_y(x, 0, t) = 2\cosh t,$$

and the initial condition

$$f(x, y, 0) = y^2 \tag{4.7}$$

Taking Ramadan Group transform to both sides of equation (4.6) we get

$$\frac{sRG[f(x, y, t)] - f(x, y, 0)}{u} = \frac{1}{2}[y^2 RG[f_{xx}(x, y, t)] + x^2 RG[f_{yy}(x, y, t)]]$$

Which can be rewritten further as

$$RG[f(x, y, t)] = \frac{y^2}{s} + \frac{u}{2s}[y^2 RG[f_{xx}(x, y, t)] + x^2 RG[f_{yy}(x, y, t)]] \tag{4.8}$$

Applying the inverse Ramadan Group transform of Eq. (4.8) implies that

$$f(x, y, t) = y^2 + RG^{-1}\{\frac{u}{2s}[y^2 RG[f_{xx}(x, y, t)] + x^2 RG[f_{yy}(x, y, t)]]$$

Using the reduced differential transform method, this leads to the recursive relation

$$f_{m+1}(x, y, t) = RG^{-1}\{\frac{u}{2s}[y^2 RG[A_m] + x^2 RG[B_m]] \quad and \quad f_0(x, y, t) = y^2 \tag{4.9}$$

where $A_m = \frac{d^2}{dx^2} f(x, y, m)$, $B_m = \frac{d^2}{dy^2} f(x, y, m)$ are the reduced differential transform of $f_{xx}(x, y, t)$ *and* $f_{yy}(x, y, t)$ respectively.

For $m = 0$, *we get* $A_0 = 0$, $B_0 = 2$, *and then we have* $f_1 = x^2 RG^{-1}[\frac{u}{s^2}] = x^2 t$

At $m = 1$, *we obtain* $A_1 = 2t$, $B_1 = 0$, *and then* $f_2 = y^2 RG^{-1}[\frac{u^2}{s^3}] = y^2 \frac{t^2}{2!}$

For $m = 2$, *we get* $A_2 = 0$, $B_2 = t^2$ *and hence* $f_3 = x^2 RG^{-1}[\frac{u^3}{s^4}] = x^2 \frac{t^3}{3!}$

For $m = 3$, *we obtain* $A_3 = \frac{t^3}{3}$, $B_3 = 0$ *and therefore* $f_4 = y^2 RG^{-1}[\frac{u^4}{s^5}] = y^2 \frac{t^4}{4!}$

And so on, then we have

$$f(x,y,t) = y^2 + x^2 t + y^2 \frac{t^2}{2!} + x^2 \frac{t^3}{3!} + y^2 \frac{t^4}{4!} + \ldots$$
$$= y^2[1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \ldots] + x^2[t + \frac{t^3}{3!} + \frac{t^5}{5!} + \ldots] = y^2 \sinh t + x^2 \cosh t$$

which is the analytical solution of equation (4.6 – 4.7).

Example 3

Consider the boundary value problem [5].

$$f_t = x^4 y^4 z^4 + \frac{1}{36}(x^2 f_{xx} + y^2 f_{yy} + z^2 f_{zz}), \qquad 0 < x, y, z < 1,\ t > 0 \tag{4.10}$$

with to Neumann boundary conditions

$$\begin{aligned} &f(0,y,z,t) = 0, && f(1,y,z,t) = y^4 z^4 (e^t - 1), \\ &f(x,0,z,t) = 0, && f(x,1,z,t) = x^4 z^4 (e^t - 1), \\ &f(x,y,0,t) = 0, && f(x,y,1,t) = x^4 z^4 (e^t - 1), \end{aligned} \tag{4.11}$$

and the initial condition

$$f(x,y,z,0) = 0\,. \tag{4.12}$$

Taking Ramadan Group transform to both sides of Eq. (4.10) we get

$$\frac{sRG[f(x,y,z,t)] - f(x,y,z,0)}{u} = \frac{1}{s}(x^4 y^4 z^4) + \frac{1}{36}[x^2 RG[f_{xx}] + y^2 RG[f_{yy}] + z^2 RG[f_{zz}]] \tag{4.13}$$

Using the initial condition the above equation can be reduced to

$$RG[f(x,y,z,t)] = \frac{u}{s^2}(x^4 y^4 z^4) + \frac{u}{36s}[x^2 RG[f_{xx}] + y^2 RG[f_{yy}] + z^2 RG[f_{zz}]]$$

Applying the inverse Ramadan Group transform of Eq. (4.11) implies that

$$f(x,y,z,t) = x^4 y^4 z^4 t + RG^{-1}\{\frac{u}{36s}[x^2 RG[f_{xx}] + y^2 RG[f_{yy}] + z^2 RG[f_{zz}]]\} \tag{4.14}$$

Applying the reduced differential transform method, the following recursive relation is obtained

$$f_0(x,y,z,t) = x^4 y^4 z^4 t,\ \ f_{m+1}(x,y,z,t) = RG^{-1}\{\frac{u}{36s}[x^2 RG[A_m] + y^2 RG[B_m] + z^2 RG[C_m]]\}$$

where $A_m = \frac{d^2}{dx^2} f(x,y,z,m)$, $B_m = \frac{d^2}{dy^2} f(x,y,z,m)$, $C_m = \frac{d^2}{dz^2} f(x,y,z,m)$ are the reduced differential transform of $f_{xx}(x,y,z,t)$, $f_{yy}(x,y,z,t)$ *and* $f_{zz}(x,y,z,t)$ respectively.

$$At \;\; m = 0 \rightarrow A_0 = 12x^2y^4z^4t \;,\; B_0 = 12x^4y^2z^4t \;\; and \;\; C_0 = 12x^4y^4z^2t$$

$$then \;\; f_1 = x^4y^4z^4 RG^{-1}[\frac{u^2}{s^3}] = x^4y^4z^4 \frac{t^2}{2!}$$

$$At \;\; m = 1 \rightarrow A_1 = 6x^2y^4z^4t^2 \;,\; B_1 = 6x^4y^2z^4t^2 \;\; and \;\; C_1 = 6x^4y^4z^2t^2$$

$$then \;\; f_2 = x^4y^4z^4 RG^{-1}[\frac{u^3}{s^4}] = x^4y^4z^4 \frac{t^3}{3!}$$

$$At \;\; m = 2 \rightarrow A_2 = 2x^2y^4z^4t^3 \;,\; B_2 = 2x^4y^2z^4t^3 \;\; and \;\; C_2 = 2x^4y^4z^2t^3$$

$$then \;\; f_3 = x^4y^4z^4 RG^{-1}[\frac{u^4}{s^5}] = x^4y^4z^4 \frac{t^4}{4!}$$

And so on, then the solution of equation (1) is the following

$$f(x,y,z,t) = x^4y^4z^4[t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + ...] = x^4y^4z^4(1 - e^t)$$

which is the exact solution of the considered problem.

Example 4

We consider the one-dimensional initial boundary value problem [5].

$$y_{tt} = \frac{1}{2} x^2 y_{xx} \;,\; 0 < x < 1 \;, t > 0, \tag{4.15}$$

Subject to Neumann boundary conditions

$$y(0,t) = 0, \qquad y(1,t) = 1 + \sinh t, \tag{4.16}$$

and the initial conditions

$$y(x,0) = x, \qquad y_t(x,0) = x^2. \tag{4.17}$$

Taking Ramadan Group transform to both sides of equation (4.15) we obtain

$$\frac{s^2 RG[y(x,t)] - sy(x,0) - uy_t(x,)}{u^2} = \frac{1}{2} x^2 RG[y_{xx}(x,t)] \,.$$

That is,

$$RG[y(x,t)] = \frac{x}{s} + \frac{u}{s^2}x^2 + \frac{u^2}{2s^2}x^2 RG[y_{xx}(x,t)] \ . \tag{4.18}$$

Applying the inverse Ramadan Group transform of Eq. (4.18) implies that

$$y(x,t) = x + x^2 t + RG^{-1}\{\frac{u^2}{2s^2}x^2 RG[y_{xx}(x,t)] \ . \tag{4.19}$$

Using the reduced differential transform method, this leads to the recursive relation

$$y_0(x,t) = x + x^2 t \quad , \qquad y_{m+1}(x,t) = RG^{-1}\{\frac{u^2}{2s^2}x^2 RG[A_m]\} \ , \tag{4.20}$$

where $A_m = \frac{d^2}{dx^2}y(x,m)$ is the reduced differential transform of $y_{xx}(x,t)$

For $m=0$, *we get* $A_0 = 2t$ *and we have* $y_1 = RG^{-1}\{\frac{u^2}{2s^2}x^2 RG[2t]\} = x^2\frac{t^3}{3!}$.

For $m=1$, *we obtain* $A_1 = 2t$, *and hence* $y_2 = RG^{-1}\{\frac{u^2}{2s^2}x^2 RG[\frac{t^3}{3}]\} = x^2\frac{t^5}{5!}$.

And so on we get,

$$y(x,t) = x + x^2 t + x^2\frac{t^3}{3!} + x^2\frac{t^5}{5!} + ... = x + x^2[t + \frac{t^3}{3!} + \frac{t^5}{5!} + ... = x + x^2\sinh t \ ,$$

which is the analytical solution of equation (4.16 – 4.18).

Example 5

Consider the boundary value problem [5],

$$f_{tt} = \frac{1}{12}(x^2 f_{xx} + y^2 f_{yy}), \quad 0 < x, y < 1 \ , t > 0, \tag{4.21}$$

subject to Neumann boundary conditions

$$\begin{aligned} &f_x(0,y,t) = 0, && f_x(1,y,t) = 4\cosh t, \\ &f_y(x,0,t) = 0, && f_y(x,0,t) = 4\sinh t, \end{aligned} \tag{4.22}$$

and the initial condition

$$f(x,y,0) = x^4 \ , \qquad f_t(x,y,0) = y^4. \tag{4.23}$$

Taking Ramadan Group transform to both sides of equation (4.21) we get

$$\frac{s^2 RG[f(x,y,t)] - sf(x,y,0) - uf_t(x,y,t)}{u^2} = \frac{1}{12}(x^2 RG[f_{xx}] + y^2 RG[f_{yy}],$$

which can be rewritten as

$$RG[f(x,y,t)] = \frac{x^4}{s} + y^4 \frac{u}{s^2} + \frac{u^2}{12s^2}\{x^2 RG[f_{xx}] + y^2 RG[f_{yy}]\}\ . \tag{4.24}$$

Applying the inverse Ramadan Group transform of Eq. (4.24) implies that

$$f(x,y,t) = x^4 + y^4 t + RG^{-1}\{\frac{u^2}{12s^2}[x^2 RG[f_{xx}] + y^2 RG[f_{yy}]]\}\ . \tag{4.25}$$

Using the reduced differential transform method, this leads to the recursive relation

$$f_0(x,y,t) = x^4 + y^4 t \quad , \quad f_{m+1}(x,y,t) = RG^{-1}\{\frac{u^2}{12s^2}[x^2 RG[A_m] + y^2 RG[B_m]]\}\ ,$$

where $A_m = \frac{d^2}{dx^2} f(x,y,m)$, $B_m = \frac{d^2}{dy^2} f(x,y,m)$ are the reduced differential transform of $f_{xx}(x,y,t)$ *and* $f_{yy}(x,y,t)$ respectively.

$$For \quad m = 0\,, we \;\; get\; A_0 = 12x^2\ ,\ B_0 = 12y^2 t\ ,$$
$$then \quad f_1 = RG^{-1}[x^4 \frac{u^2}{s^3} + \frac{u^3}{s^4} y^4] = x^4 \frac{t^2}{2!} + y^4 \frac{t^3}{3!}\,.$$

$$For \quad m = 1\,, we\; obtain\; A_1 = 6x^2 t^2\ ,\ B_1 = 2y^2 t^3,$$
$$then \;\; f_2 = RG^{-1}[x^4 \frac{u^4}{s^5} + y^4 \frac{u^5}{s^6}] = x^4 \frac{t^4}{4!} + y^4 \frac{t^5}{5!}\,.$$

And so on, then the solution of equation (4.21 – 4.23) is the following

$$\begin{aligned} f(x,y,t) &= x^4 + y^4 t + x^4 \frac{t^2}{2!} + y^4 \frac{t^3}{3!} + x^4 \frac{t^4}{4!} + y^4 \frac{t^5}{5!} + \ldots \\ &= x^4[1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \ldots] + y^4[t + \frac{t^3}{3!} + \frac{t^5}{5!} + \ldots] = x^4 \cosh t + y^4 \sinh t\,. \end{aligned}$$

This is the exact solution of the considered problem.

Example 6

Consider the three-dimensional inhomogeneous initial and boundary value problem [5].

$$f_{tt} = (x^2 + y^2 + z^2) + \frac{1}{2}(x^2 f_{xx} + y^2 f_{yy} + z^2 f_{zz}), \quad 0 < x,y,z < 1\ ,\ t > 0, \tag{4.26}$$

Subject to boundary conditions

$$f(0,y,z,t)=y^2(e^t-1)+z^2(e^{-t}-1),\qquad f(1,y,z,t)=(1+y^2)(e^t-1)+z^2(e^{-t}-1),$$
$$f(x,0,z,t)=x^2(e^t-1)+z^2(e^{-t}-1),\qquad f(x,1,z,t)=(1+x^2)(e^t-1)+z^2(e^{-t}-1),\tag{4.27}$$
$$f(x,y,0,t)=(x^2+y^2)(e^{-t}-1),\qquad f(x,y,1,t)=(x^2+y^2)(e^t-1)+(e^{-t}-1),$$

and the initial condition

$$f(x,y,z,0)=0\ ,\qquad f_t(x,y,z,0)=x^2+y^2-z^2.\tag{4.28}$$

Taking Ramadan Group transform to both sides of equation (4.26) we get

$$\frac{s^2RG[f(x,y,z,t)]-sf(x,y,z,0)-uf_t(x,y,z,t)}{u^2}=\frac{(x^2+y^2+z^2)}{s}+\frac{1}{2}(x^2RG[f_{xx}]$$
$$+y^2RG[f_{yy}]+z^2RG[f_{zz}])$$

$$RG[f(x,y,z,t)]=\frac{u}{s^2}(x^2+y^2-z^2)+\frac{u^2}{s^3}(x^2+y^2+z^2)+$$
$$\frac{u^2}{2s^2}[x^2RG[f_{xx}]+y^2RG[f_{yy}]+z^2RG[f_{zz}]]\tag{4.29}$$

Applying the inverse Ramadan Group transform of Eq. (4.29) implies that

Using the reduced differential transform method, this leads to the recursive relation

$$f_0(x,y,z,t)=(x^2+y^2-z^2)t\quad ,\ f_1=(x^2+y^2+z^2)\frac{t^2}{2!}$$
$$f_{m+2}(x,y,z,t)=RG^{-1}\{\frac{u^2}{2s^2}[x^2RG[A_m]+y^2RG[B_m]+z^2RG[C_m]]\}\tag{4.30}$$

Where $A_m=\frac{d^2}{dx^2}f(x,y,z,m)$, $B_m=\frac{d^2}{dy^2}f(x,y,z,m)$, $C_m=-\frac{d^2}{dz^2}f(x,y,z,m)$ are the reduced differential transforms of f_{xx}, f_{yy} *and* f_{zz}.

$$At\ \ m=0\rightarrow A_0=2t,\ \ B_0=2t,\ \ C_0=-2t\ ,$$
$$f_2=RG^{-1}\{\frac{u^2}{2s^2}RG[2t](x^2+y^2-z^2)\}=(x^2+y^2-z^2)\frac{t^3}{3!}$$

$$At\ \ m=1\rightarrow A_1=t^2,\ \ B_1=t^2,\ \ C_1=t^2\ ,$$
$$f_3=RG^{-1}\{\frac{u^2}{2s^2}RG[t^2](x^2+y^2+z^2)\}=(x^2+y^2+z^2)\frac{t^4}{4!}.$$

And so on, then the solution of equation (4.26 – 4.28) is the following

$$f(x,y,z,t)=(x^2+y^2-z^2)[t+\frac{t^3}{3!}+...]+(x^2+y^2+z^2)[\frac{t^2}{2!}+\frac{t^4}{4!}+...],$$

$$f(x,y,z,t)=(x^2+y^2)[t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+...]+z^2[-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}+...],$$

Then the exact solution is

$$f(x,y,z,t)=(x^2+y^2)(e^t-1)+z^2(e^{-t}-1).$$

Example 7

Consider the linear Klein-Gordon equation in the form [6].

$$y_{tt}(x,t)-y_{xx}(x,t)-y(x,t)=0, \tag{4.31}$$

with the initial conditions

$$y(x,0)=1+\sin x\ ,\ \ y_t(x,0)=0\ . \tag{4.32}$$

Taking Ramadan Group transform to both sides of equation (1).

$$\frac{s^2RG[y(x,t)]-sy(x,0)-uy_t(x,0)}{u^2}=RG[y_{xx}(x,t)+y(x,t)],$$

$$RG[y(x,t)]=\frac{1}{s}(1+\sin x)+\frac{u^2}{s^2}[RG[y_{xx}+y]]. \tag{4.33}$$

Applying the inverse Ramadan Group transform of Eq. (4.31) implies that

$$y(x,t)=1+\sin x+RG^{-1}\{\frac{u^2}{s^2}[RG[y_{xx}+y]]\}\ .$$

Using the reduced differential transform method, this leads to the recursive relation.

$$y_0(x,t)=1+\sin x\quad ,\quad y_{m+1}(x,t)=RG^{-1}\{\frac{u^2}{s^2}RG[A_m+B_m]\}, \tag{4.34}$$

where $A_m=\frac{d^2}{dx^2}y(x,m)$ *and* $B_m=y(x,m)$ are reduced differential transforms of $y_{xx}(x,t)$ *and* $y(x,t)$ respectively.

For $m=0$ *we get* $A_0=-\sin x$, $B_0=1+\sin x$,

which implies that $y_1=RG^{-1}[\frac{u^2}{s^2}RG[-\sin x+1+\sin x]]=\frac{t^2}{2!}.$

$$Also,\ for\ \ m=1\ we\ get\ A_1=0,\ B_1=\frac{t^2}{2!},$$

$$and\ hence\ we\ get\ \ y_2=RG^{-1}[\frac{u^2}{s^2}RG[\frac{t^2}{2!}]]=RG^{-1}[\frac{u^4}{s^5}]=\frac{t^4}{4!},$$

$$Similarly\ ,\ for\ \ m=2\ we\ have\ A_2=0,\ B_2=\frac{t^4}{4!},$$

$$and\ then\ we\ have\ \ y_3=RG^{-1}[\frac{u^2}{s^2}RG[\frac{t^4}{4!}]]=RG^{-1}[\frac{u^6}{s^7}]=\frac{t^6}{6!}.$$

And so on, then the solution of equation (4.31, 4.32) is the following

$$y(x,t)=1+\sin x+\frac{t^2}{2!}+\frac{t^4}{4!}+\frac{t^6}{6!}+...=\sin x+\cosh t,$$

which is the exact solution for the considered problem.

Example 8

Consider the nonlinear partial differential equation

$$y_t(x,t)+y_x(x,t)+u^2(x,t)=0\ , \tag{4.35}$$

with the initial condition

$$y(x,0)=\frac{1}{2x}. \tag{4.36}$$

Taking Ramadan Group transform to both sides of equation (4.35) we obtain

$$\frac{sRG[y(x,t)]-y(x,0)}{u}=-RG[y_x(x,t)+y^2(x,t)].$$

That is,

$$RG[y(x,t)]=\frac{1}{2sx}-\frac{u}{s}RG[y_x+y^2]\ . \tag{4.37}$$

Applying the inverse Ramadan Group transform of Eq. (4.35) implies that

$$y(x,t)=\frac{1}{2x}-RG^{-1}\{\frac{u}{s}[RG[y_x+y^2]]\}. \tag{4.38}$$

Using the reduced differential transform method, this leads to the recursive relation.

$$y_0(x,t)=\frac{1}{2x}\quad ,\qquad y_{m+1}(x,t)=-RG^{-1}\{\frac{u}{s}RG[A_m+B_m]\}, \tag{4.39}$$

where $A_m = \frac{d}{dx} y(x,m)$ *and* $B_m = \sum_{r=0}^{m} y_r(x) y_{m-r}(x)$ are reduced differential transforms of $y_x(x,t)$ *and* $y^2(x,t)$.

For $m=0$ *we get* $A_0 = -\frac{1}{2x^2}$, $B_0 = \frac{1}{4x^2}$ *which implies that* $y_1 = -RG^{-1}[\frac{u}{s} RG[\frac{-1}{4x^2}]] = \frac{t}{4x^2}$.

Also for $m=1$ *wehave* $A_1 = \frac{-t}{2x^3}$, $B_1 = \frac{t}{4x^3}$ *and hence we get* $y_2 = -RG^{-1}[\frac{u}{s} RG[\frac{-t}{4x^3}]] = \frac{t^2}{8x^3}$.

For $m=2$ *we obtain* $A_2 = \frac{-3t^2}{8x^3}$, $B_2 = \frac{3t^2}{16x^4}$,
and therefore we have $y_3 = -RG^{-1}[\frac{u}{s} RG[\frac{-3t^2}{16x^4}]] = \frac{t^3}{16x^4}$.

And so on, then the solution of equation (4.35- 4.36) is the following

$$y(x,t) = \frac{1}{2x} + \frac{t}{4x^2} + \frac{t^2}{8x^3} + \frac{t^3}{16x^4} + \ldots = \frac{1}{2x}(1 + \frac{t}{2x} + \frac{t^2}{4x^2} + \frac{t^3}{8x^3} + \ldots) = \frac{1}{2x-t} ,$$

which is the analytical solution of the considered problem.

Remark: Behaviour of parameters of Ramadan Group Transform

Ramadan Group Transform is a generalization of both Laplace and Sumudu transforms: if $u=1$, the transform is reduced to Laplace transform and if $s=1$, it reduces to Sumudu transform.

5 Analysis of the Proposed Hybrid Method

The method is simple, effective, efficient and easy to use where the main benefit of it is to offer the analytical approximation. Using this proposed method in many cases the exact solution can be obtained in a rapid convergent series. The method is much simpler than other similar methods as Laplace- Adomian or Sumudu – Adomian methods.

6 Concluding Remarks

A combined form of Ramadan group integral transform with the reduced differential transform is effectively used to handle eight examples of nonlinear PDEs. The exact solution has been obtained even with just the first few terms, which indicates that the proposed method needs much less computational work. The proposed scheme can be applied for other nonlinear PDEs.

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Competing Interests

Authors have declared that no competing interests exist.

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