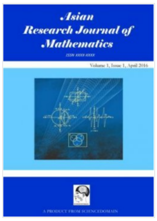

Asian Research Journal of Mathematics

6(1): 1-19, 2017; Article no.ARJOM.35481

ISSN: 2456-477X

The 19 Densities of the Hierarchical Bayes Model with Two Conditional Levels

Ying-Ying Zhang[1*], Teng-Zhong Rong[1] and Man-Man Li[1]

[1] *Department of Statistics and Actuarial Science, College of Mathematics and Statistics, Chongqing University, Chongqing, China.*

Authors' contributions

This work was carried out in collaboration between all authors. Author YYZ proved the theorem and did the Bayesian example. Author TZR did literature searches and revised the manuscript. Author MML revised the manuscript. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/ARJOM/2017/35481

Editor(s):

(1) Hari Mohan Srivastava, Department of Mathematics and Statistics, University of Victoria, Canada.

(2) William Chin, DePaul University, Chicago, USA.

Reviewers:

(1) G. Y. Sheu, Chang-Jung Christian University, Tainan, Taiwan.

(2) Thomas L. Toulias, Technological Educational Institute of Athens, Greece.

Complete Peer review History: http://www.sciencedomain.org/review-history/20812

Received: ***15th July 2017***

Accepted: ***30th August 2017***

Published: ***4th September 2017***

Original Research Article

Abstract

There are 19 densities involved in the hierarchical Bayes model with two conditional levels, in which the 3 densities, that is, the likelihood function, the first level prior density, and the second level prior density, are known densities. We have written the 16 unknown densities in terms of the 3 known densities in a theorem which is very handy for practitioners and researchers interested in the hierarchical Bayes model with two conditional levels. Finally, we apply the theorem to a specific hierarchical normal Bayes model with two conditional levels and obtain the functional forms of the 16 unknown densities. Moreover, we figure out the exact distributions of the 16 densities, which are one-, two-, or three-dimensional normal distributions.

Keywords: *Hierarchical Bayes model; hierarchical Bayes analysis; two conditional levels; Bayesian analysis; densities.*

2010 Mathematics Subject Classification: 62C10, 62F15.

**Corresponding author: E-mail: robertzhang@cqu.edu.cn, robertzhangyying@qq.com;*

1 Introduction

Bayesian approaches are continually developing, with [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] being some of the most important works. There is an ambivalent aspect of Bayesian analysis: It is sufficiently reductive to produce an effective decision, but this efficiency can also be misused. A pertinent criticism is that the prior information is rarely rich enough to define a prior distribution exactly. The empirical Bayes analysis, see [13, 14, 15, 16, 17, 4, 18, 19, 20] among others, is based on a perception of imprecision over the prior information, but at a more pragmatic level. The empirical Bayes analysis relies on a conjugate prior modeling, where the hyperparameters are estimated from the observations and this "estimated prior" is then used as a regular prior in the subsequent inference. However, the empirical Bayes analysis is out of the Bayesian paradigm. Alternatively, the hierarchical Bayes analysis (see [21, 22, 23, 4, 24, 8, 12]) considers that the imprecision over the prior information can be done within the Bayesian paradigm, according to which, uncertainty at any level is incorporated into prior distributions. In the simplest cases, the hierarchical structure is reduced to two prior levels. The first level (or lower level) prior distribution is generally a conjugate prior, owing to the computational tractability of these distributions. The second level (or upper level) prior distribution is usually a noninformative prior due to lack of information. The hierarchical Bayes modeling has many applications in real life, such as medicine, biology, animal breeding, economics, and so on. In meta-analysis, several experiments about the same phenomenon undertaken at different places with different subjects and different protocols are pooled together (see [25, 26]).

The author in [8] has listed several justifications for the hierarchical Bayes analysis. It is also pointed out that it is seldom necessary to go beyond two conditional levels in the hierarchical decomposition. For the hierarchical Bayes model with two conditional levels, Lemma 10.2.9 in [[8], pg.466] has calculated $\pi(\theta|x)$ in terms of $\pi(\theta|x,\theta_1)$ and $\pi(\theta_1|x)$, which in turn depend on the 3 known densities $\pi(x|\theta)$, $\pi(\theta|\theta_1)$, and $\pi(\theta_1)$. In fact, there are 19 densities involved in the hierarchical Bayes model with two conditional levels, and they can be concisely written in Fig. 1. Inspired by the lemma, we have calculated the remaining 16 densities in terms of the 3 known densities, and the result is summarized in Theorem 2.1.

The rest of the paper is organized as follows. In the next Section 2, we have written the 16 unknown densities in terms of the 3 known densities in Theorem 2.1. In Section 3, we apply Theorem 2.1 to a hierarchical normal Bayes model with two conditional levels and obtain the functional forms of the 16 densities. Moreover, we figure out the exact distributions of the 16 densities. Section 4 concludes.

2 Main Results

We consider the following hierarchical Bayes model with two conditional levels:

$$\begin{cases} x|\theta \sim \pi(x|\theta), \\ \theta|\theta_1 \sim \pi(\theta|\theta_1), \\ \theta_1 \sim \pi(\theta_1). \end{cases}$$

In [8], the 3 densities are written as $\pi(x|\theta) = f(x|\theta)$, $\pi(\theta|\theta_1) = \pi_1(\theta|\theta_1)$, and $\pi(\theta_1) = \pi_2(\theta_1)$. Here, we intentionally write all the densities as $\pi()$ to lighten notations and also to focus on the arguments. Let $x \in \mathcal{X}$, $\theta \in \Theta$, and $\theta_1 \in \Theta_1$. We will use $\int f(x)\,dx$, $\int g(\theta)\,d\theta$, and $\int h(\theta_1)\,d\theta_1$ to represent $\int_{\mathcal{X}} f(x)\,dx$, $\int_{\Theta} g(\theta)\,d\theta$, and $\int_{\Theta_1} h(\theta_1)\,d\theta_1$, respectively, that is, we omit the domain of integration to lighten notations. The 19 densities involved in the hierarchical Bayes model with two conditional levels, can be concisely presented in Fig. 1.

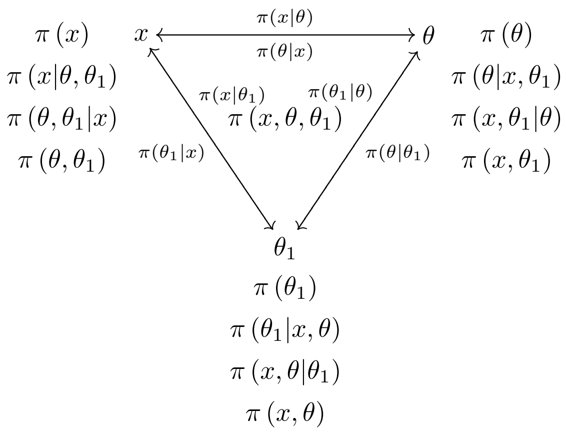

Fig. 1. The 19 densities of the hierarchical Bayes model with two conditional levels

In the hierarchical Bayes model with two conditional levels, we usually assume that the 3 densities $\pi(x|\theta)$, $\pi(\theta|\theta_1)$, and $\pi(\theta_1)$ are known densities. Our goal is to write the other 16 densities in terms of the 3 known densities.

We have the following lemma which states an equivalence relationship between two equations.
Lemma 2.1

$$\pi(x|\theta, \theta_1) = \pi(x|\theta) \tag{2.1}$$

is equivalent to

$$\pi(x, \theta, \theta_1) = \pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1). \tag{2.2}$$

Proof. Assume that relation (2.1) holds. Then

$$\pi(x, \theta, \theta_1) = \pi(x|\theta, \theta_1)\pi(\theta, \theta_1) = \pi(x|\theta)\pi(\theta|\theta_1)\pi(\theta_1).$$

Conversely, if relation (2.2) holds, we derive that

$$\pi(x|\theta, \theta_1)\pi(\theta, \theta_1) = \pi(x, \theta, \theta_1) = \pi(x|\theta)\pi(\theta|\theta_1)\pi(\theta_1) = \pi(x|\theta)\pi(\theta, \theta_1),$$

and thus $\pi(x|\theta, \theta_1) = \pi(x|\theta)$. □

To calculate the other 16 densities in terms of the 3 known densities, we make the following assumptions.

(A1). (2.1) or (2.2) holds true.

(A2). All the 19 densities are positive proper densities, that is, they are positive and integrate to 1.

(A3). $(x, \theta, \theta_1) \in \mathcal{X} \times \Theta \times \Theta_1$, so that changing the order of integration is allowed.

With the preparations of Lemma 2.1 and the three assumptions, we have the following theorem, in which we have written the 16 unknown densities in terms of the 3 known densities.

Theorem 2.1 *Let the assumptions (A1), (A2), and (A3) hold. Then we can calculate the other 16 densities in terms of the 3 known densities* $\pi(x|\theta)$, $\pi(\theta|\theta_1)$, *and* $\pi(\theta_1)$ *as follows. The following 5 densities are related to* x*.*

$$\pi(x) = \int\int \pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1)\,d\theta d\theta_1,$$

$$\pi\left(x|\theta_1\right) = \int \pi\left(x|\theta\right)\pi\left(\theta|\theta_1\right)d\theta,$$

$$\pi\left(x|\theta,\theta_1\right) = \pi\left(x|\theta\right),$$

$$\pi\left(\theta,\theta_1|x\right) = \frac{\pi\left(x|\theta\right)\pi\left(\theta|\theta_1\right)\pi\left(\theta_1\right)}{\int\int \pi\left(x|\theta\right)\pi\left(\theta|\theta_1\right)\pi\left(\theta_1\right)d\theta d\theta_1} \propto \pi\left(x|\theta\right)\pi\left(\theta|\theta_1\right)\pi\left(\theta_1\right),$$

$$\pi\left(\theta,\theta_1\right) = \pi\left(\theta|\theta_1\right)\pi\left(\theta_1\right).$$

The following 5 densities are related to θ.

$$\pi\left(\theta\right) = \int \pi\left(\theta|\theta_1\right)\pi\left(\theta_1\right)d\theta_1,$$

$$\pi\left(\theta|x\right) = \frac{\int \pi\left(x|\theta\right)\pi\left(\theta|\theta_1\right)\pi\left(\theta_1\right)d\theta_1}{\int\int \pi\left(x|\theta\right)\pi\left(\theta|\theta_1\right)\pi\left(\theta_1\right)d\theta d\theta_1} \propto \int \pi\left(x|\theta\right)\pi\left(\theta|\theta_1\right)\pi\left(\theta_1\right)d\theta_1,$$

$$\pi\left(\theta|x,\theta_1\right) = \frac{\pi\left(x|\theta\right)\pi\left(\theta|\theta_1\right)}{\int \pi\left(x|\theta\right)\pi\left(\theta|\theta_1\right)d\theta} \propto \pi\left(x|\theta\right)\pi\left(\theta|\theta_1\right),$$

$$\pi\left(x,\theta_1|\theta\right) = \frac{\pi\left(x|\theta\right)\pi\left(\theta|\theta_1\right)\pi\left(\theta_1\right)}{\int \pi\left(\theta|\theta_1\right)\pi\left(\theta_1\right)d\theta_1} \propto \pi\left(x|\theta\right)\pi\left(\theta|\theta_1\right)\pi\left(\theta_1\right),$$

$$\pi\left(x,\theta_1\right) = \int \pi\left(x|\theta\right)\pi\left(\theta|\theta_1\right)\pi\left(\theta_1\right)d\theta.$$

The following 5 densities are related to θ_1.

$$\pi\left(\theta_1|x\right) = \frac{\int \pi\left(x|\theta\right)\pi\left(\theta|\theta_1\right)\pi\left(\theta_1\right)d\theta}{\int\int \pi\left(x|\theta\right)\pi\left(\theta|\theta_1\right)\pi\left(\theta_1\right)d\theta d\theta_1} \propto \int \pi\left(x|\theta\right)\pi\left(\theta|\theta_1\right)\pi\left(\theta_1\right)d\theta,$$

$$\pi\left(\theta_1|\theta\right) = \frac{\pi\left(\theta|\theta_1\right)\pi\left(\theta_1\right)}{\int \pi\left(\theta|\theta_1\right)\pi\left(\theta_1\right)d\theta_1} \propto \pi\left(\theta|\theta_1\right)\pi\left(\theta_1\right),$$

$$\pi\left(\theta_1|x,\theta\right) = \pi\left(\theta_1|\theta\right) = \frac{\pi\left(\theta|\theta_1\right)\pi\left(\theta_1\right)}{\int \pi\left(\theta|\theta_1\right)\pi\left(\theta_1\right)d\theta_1} \propto \pi\left(\theta|\theta_1\right)\pi\left(\theta_1\right),$$

$$\pi\left(x,\theta|\theta_1\right) = \pi\left(x|\theta\right)\pi\left(\theta|\theta_1\right),$$

$$\pi\left(x,\theta\right) = \int \pi\left(x|\theta\right)\pi\left(\theta|\theta_1\right)\pi\left(\theta_1\right)d\theta_1.$$

Finally, the joint density

$$\pi\left(x,\theta,\theta_1\right) = \pi\left(x|\theta\right)\pi\left(\theta|\theta_1\right)\pi\left(\theta_1\right).$$

Note that in the above theorem the observation x can be replaced by the sample $\boldsymbol{x}$. If the random variables x, θ, or θ_1 are discrete, then the integrals can be replaced by the sums.

Proof. Note that 4 of the 16 densities are obviously represented by the 3 known densities. They are $\pi\left(x|\theta,\theta_1\right)$, $\pi\left(\theta,\theta_1\right)$, $\pi\left(\theta\right)$, and $\pi\left(x,\theta,\theta_1\right)$. Apart from the 4 obvious densities, the other 12 densities need to be calculated. We find that some of the 12 densities can be calculated by the 3 known densities and the 4 obvious densities, and they should be calculated first. So their calculation order is 1, and we refer them to order 1 densities. Some of the remaining densities depend on the order 1 densities, and we refer them to order 2 densities. Finally, order i densities depend on order $i-1$ densities for $i = 2,3,4,5$. The 12 densities, their dependence densities, and the order of calculation is summarized in Table 1. In the table, M_i, $i = 1,2,3$, represents Method i, and O_i, $i = 1,2,3,4,5$, represents Order i. In Table 1, note that $\pi\left(\theta,\theta_1|x\right)$ can be calculated by three methods. By method 1, $\pi\left(\theta,\theta_1|x\right)$ depends on $\pi\left(x\right)$ which is an order 3 density, so in this case $\pi\left(\theta,\theta_1|x\right)$ is called an order 4 density. By method 2, $\pi\left(\theta,\theta_1|x\right)$ depends on $\pi\left(\theta|x,\theta_1\right)$ and $\pi\left(\theta_1|x\right)$, which are order 3 and order 4 densities, so in this case $\pi(\theta, \theta_1|x)$ is called an order 5 density. By method 3, $\pi(\theta, \theta_1|x)$ depends on $\pi(\theta_1|x, \theta)$ and $\pi(\theta|x)$, which are order 1 and order 4 densities, so in this case $\pi(\theta, \theta_1|x)$ is called an order 5 density. We call $\pi(\theta, \theta_1|x)$ an order 5 density because we use the highest order of the density of the three methods. The order of the density is useful only to facilitate the calculations by orders.

Table 1. The 12 densities, their dependence densities, and the order of calculation

Target densities	Dependence densities	Order of calculation
$\pi(x)$	M_1: $\pi(\theta)$; M_2: $\pi(x|\theta_1)$ (O_2)	O_3
$\pi(x|\theta_1)$	$\pi(x, \theta|\theta_1)$ (O_1)	O_2
$\pi(\theta, \theta_1|x)$	M_1: $\pi(x)$ (O_3); M_2: $\pi(\theta|x, \theta_1)$ (O_3), $\pi(\theta_1|x)$ (O_4); M_3: $\pi(\theta_1|x, \theta)$ (O_1), $\pi(\theta|x)$ (O_4)	O_5
$\pi(\theta|x)$	$\pi(\theta)$, $\pi(x)$ (O_3)	O_4
$\pi(\theta|x, \theta_1)$	$\pi(x|\theta_1)$ (O_2)	O_3
$\pi(x, \theta_1|\theta)$	M_1: $\pi(\theta)$; M_2: $\pi(\theta_1|\theta)$ (O_1)	O_2
$\pi(x, \theta_1)$	$\pi(x|\theta_1)$ (O_2)	O_3
$\pi(\theta_1|x)$	$\pi(x)$ (O_3), $\pi(x|\theta_1)$ (O_2)	O_4
$\pi(\theta_1|\theta)$		O_1
$\pi(\theta_1|x, \theta)$		O_1
$\pi(x, \theta|\theta_1)$		O_1
$\pi(x, \theta)$		O_1

Now we calculate the order 1 densities $\pi(\theta_1|\theta)$, $\pi(\theta_1|x, \theta)$, $\pi(x, \theta|\theta_1)$, and $\pi(x, \theta)$ sequentially. It is easy to show that

$$\pi(\theta_1|\theta) = \frac{\pi(\theta|\theta_1)\,\pi(\theta_1)}{\pi(\theta)} = \frac{\pi(\theta|\theta_1)\,\pi(\theta_1)}{\int \pi(\theta|\theta_1)\,\pi(\theta_1)\,d\theta_1} \propto \pi(\theta|\theta_1)\,\pi(\theta_1).$$

For $\pi(\theta_1|x, \theta)$, we have

$$\pi(\theta_1|x, \theta)\,\pi(x|\theta)\,\pi(\theta) = \pi(\theta_1|x, \theta)\,\pi(x, \theta) = \pi(x, \theta, \theta_1) = \pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1),$$

and therefore,

$$\pi(\theta_1|x, \theta) = \frac{\pi(\theta|\theta_1)\,\pi(\theta_1)}{\pi(\theta)} = \pi(\theta_1|\theta) = \frac{\pi(\theta|\theta_1)\,\pi(\theta_1)}{\int \pi(\theta|\theta_1)\,\pi(\theta_1)\,d\theta_1} \propto \pi(\theta|\theta_1)\,\pi(\theta_1).$$

For $\pi(x, \theta|\theta_1)$, we have

$$\pi(x, \theta|\theta_1)\,\pi(\theta_1) = \pi(x, \theta, \theta_1) = \pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1),$$

and thus,

$$\pi(x, \theta|\theta_1) = \pi(x|\theta)\,\pi(\theta|\theta_1).$$

It is easy to show that

$$\pi(x, \theta) = \pi(x|\theta)\,\pi(\theta) = \pi(x|\theta)\int \pi(\theta|\theta_1)\,\pi(\theta_1)\,d\theta_1 = \int \pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1)\,d\theta_1.$$

After that, we calculate the order 2 densities $\pi(x|\theta_1)$ and $\pi(x, \theta_1|\theta)$ sequentially. The density $\pi(x|\theta_1)$ depends on $\pi(x, \theta|\theta_1)$, and thus

$$\pi(x|\theta_1) = \int \pi(x, \theta|\theta_1)\,d\theta = \int \pi(x|\theta)\,\pi(\theta|\theta_1)\,d\theta.$$

The density $\pi(x,\theta_1|\theta)$ can be calculated by two methods. Method 1: We have

$$\pi(x,\theta_1|\theta)\,\pi(\theta) = \pi(x,\theta,\theta_1) = \pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1),$$

and thus

$$\pi(x,\theta_1|\theta) = \frac{\pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1)}{\pi(\theta)} = \frac{\pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1)}{\int \pi(\theta|\theta_1)\,\pi(\theta_1)\,d\theta_1} \propto \pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1).$$

Method 2: As before, we have

$$\begin{aligned}\pi(x,\theta_1|\theta) &= \frac{\pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1)}{\pi(\theta)} = \frac{\pi(x|\theta)\,\pi(\theta_1|\theta)\,\pi(\theta)}{\pi(\theta)} = \pi(x|\theta)\,\pi(\theta_1|\theta)\\ &= \frac{\pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1)}{\int \pi(\theta|\theta_1)\,\pi(\theta_1)\,d\theta_1} \propto \pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1).\end{aligned}$$

Next, we calculate the order 3 densities $\pi(x)$, $\pi(\theta|x,\theta_1)$, and $\pi(x,\theta_1)$ sequentially. The density $\pi(x)$ can be calculated by two methods. Method 1 is by exploiting the expression of $\pi(\theta)$, so

$$\begin{aligned}\pi(x) &= \int \pi(x|\theta)\,\pi(\theta)\,d\theta = \int \pi(x|\theta)\int \pi(\theta|\theta_1)\,\pi(\theta_1)\,d\theta_1 d\theta\\ &= \int\int \pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1)\,d\theta_1 d\theta = \int\int \pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1)\,d\theta d\theta_1.\end{aligned}$$

Method 2 is by exploiting the expression of $\pi(x|\theta_1)$, so

$$\pi(x) = \int \pi(x|\theta_1)\,\pi(\theta_1)\,d\theta_1 = \int\int \pi(x|\theta)\,\pi(\theta|\theta_1)\,d\theta\pi(\theta_1)\,d\theta_1 = \int\int \pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1)\,d\theta d\theta_1.$$

For $\pi(\theta|x,\theta_1)$, we have

$$\pi(\theta|x,\theta_1)\,\pi(x|\theta_1)\,\pi(\theta_1) = \pi(\theta|x,\theta_1)\,\pi(x,\theta_1) = \pi(x,\theta,\theta_1) = \pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1),$$

and thus

$$\pi(\theta|x,\theta_1) = \frac{\pi(x|\theta)\,\pi(\theta|\theta_1)}{\pi(x|\theta_1)} = \frac{\pi(x|\theta)\,\pi(\theta|\theta_1)}{\int \pi(x|\theta)\,\pi(\theta|\theta_1)\,d\theta} \propto \pi(x|\theta)\,\pi(\theta|\theta_1).$$

For $\pi(x,\theta_1)$, we have

$$\pi(x,\theta_1) = \pi(x|\theta_1)\,\pi(\theta_1) = \int \pi(x|\theta)\,\pi(\theta|\theta_1)\,d\theta\pi(\theta_1) = \int \pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1)\,d\theta.$$

Then, we calculate the order 4 densities $\pi(\theta|x)$ and $\pi(\theta_1|x)$ sequentially. For $\pi(\theta|x)$, we have

$$\begin{aligned}\pi(\theta|x) &= \frac{\pi(x|\theta)\,\pi(\theta)}{\pi(x)} = \frac{\pi(x|\theta)\int \pi(\theta|\theta_1)\,\pi(\theta_1)\,d\theta_1}{\int\int \pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1)\,d\theta d\theta_1}\\ &= \frac{\int \pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1)\,d\theta_1}{\int\int \pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1)\,d\theta d\theta_1} \propto \int \pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1)\,d\theta_1.\end{aligned}$$

For $\pi(\theta_1|x)$, we have

$$\begin{aligned}\pi(\theta_1|x) &= \frac{\pi(x|\theta_1)\,\pi(\theta_1)}{\pi(x)} = \frac{\int \pi(x|\theta)\,\pi(\theta|\theta_1)\,d\theta\pi(\theta_1)}{\int\int \pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1)\,d\theta d\theta_1}\\ &= \frac{\int \pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1)\,d\theta}{\int\int \pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1)\,d\theta d\theta_1} \propto \int \pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1)\,d\theta.\end{aligned}$$

Finally, we calculate the order 5 density $\pi(\theta,\theta_1|x)$ by three methods. Method 1: We have

$$\pi(\theta,\theta_1|x)\,\pi(x) = \pi(x,\theta,\theta_1) = \pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1),$$

and thus

$$\pi(\theta,\theta_1|x)=\frac{\pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1)}{\pi(x)}=\frac{\pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1)}{\int\int\pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1)\,d\theta d\theta_1}\propto\pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1).$$

Method 2: We have

$$\begin{aligned}\pi(\theta,\theta_1|x)=\pi(\theta|x,\theta_1)\,\pi(\theta_1|x)&=\frac{\pi(x|\theta)\,\pi(\theta|\theta_1)}{\int\pi(x|\theta)\,\pi(\theta|\theta_1)\,d\theta}\frac{\int\pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1)\,d\theta}{\int\int\pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1)\,d\theta d\theta_1}\\&=\frac{\pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1)}{\int\int\pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1)\,d\theta d\theta_1}\propto\pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1).\end{aligned}$$

Method 3: We have

$$\begin{aligned}\pi(\theta,\theta_1|x)=\pi(\theta_1|x,\theta)\,\pi(\theta|x)&=\frac{\pi(\theta|\theta_1)\,\pi(\theta_1)}{\int\pi(\theta|\theta_1)\,\pi(\theta_1)\,d\theta_1}\frac{\int\pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1)\,d\theta_1}{\int\int\pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1)\,d\theta d\theta_1}\\&=\frac{\pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1)}{\int\int\pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1)\,d\theta d\theta_1}\propto\pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1).\end{aligned}$$

The proof is complete. □

3 An Example

In this section, we will provide an example to illustrate the usage of Theorem 2.1. We consider the following hierarchical normal Bayes model with two conditional levels:

$$\begin{cases}\pi(x|\theta)\sim N(\theta,1),\\ \pi(\theta|\theta_1)\sim N(\theta_1,1),\\ \pi(\theta_1)\sim N(0,1).\end{cases}\tag{3.1}$$

Therefore,

$$\begin{aligned}\pi(x|\theta)&=\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{(x-\theta)^2}{2}\right],\\ \pi(\theta|\theta_1)&=\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{(\theta-\theta_1)^2}{2}\right],\\ \pi(\theta_1)&=\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{\theta_1^2}{2}\right].\end{aligned}$$

As described in Theorem 2.1. Let the assumptions (A1), (A2), and (A3) hold. Then we can calculate the 16 densities in terms of the 3 known densities $\pi(x|\theta)$, $\pi(\theta|\theta_1)$, and $\pi(\theta_1)$ as follows. In general, we can only obtain the functional forms of the 16 densities. However, for the simple hierarchical normal Bayes model, we can figure out the exact distributions of the 16 densities. They are one-, two-, or three-dimensional normal distributions.

Before calculating the 16 densities, we provide a standard Bayesian calculus tool, that is,

$$\begin{cases}y_m|\theta\sim N\left(\theta,\frac{\sigma^2}{m}\right),\\ \theta\sim N\left(\mu_0,\frac{\sigma^2}{n_0}\right),\end{cases}\Longrightarrow\begin{cases}\theta|y_m\sim N\left(\frac{n_0\mu_0+my_m}{n_0+m},\frac{\sigma^2}{n_0+m}\right),\\ y_m\sim N\left(\mu_0,\sigma^2\left(\frac{1}{n_0}+\frac{1}{m}\right)\right),\end{cases}\tag{3.2}$$

where μ_0, n_0, and σ^2 are known.

We first calculate the 5 densities related to x. Since x, θ, and θ_1 are normal, the integrals in this section are from $-\infty$ to ∞. We will omit the integration limits to lighten notations. We have

$$\begin{aligned}\pi(x) &= \int\int \pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1)\,d\theta d\theta_1 \\ &= \int\int \pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1)\,d\theta_1 d\theta \\ &= \int \pi(x|\theta)\left[\int \pi(\theta|\theta_1)\,\pi(\theta_1)\,d\theta_1\right] d\theta \\ &= \int \pi(x|\theta)\,\pi(\theta)\,d\theta.\end{aligned}$$

That is, $\pi(x)$ is the marginal distribution of $\pi(x|\theta)$ and $\pi(\theta)$. From (3.1) and (3.2), we can easily obtain

$$\begin{cases} \pi(x|\theta) \sim N(\theta,1), \\ \pi(\theta) \sim N(0,2). \end{cases}$$

Moreover, by (3.2), we have

$$\pi(x) \sim N(0,3).$$

Therefore,

$$\pi(x) = \frac{1}{\sqrt{2\pi}\sqrt{3}} \exp\left(-\frac{x^2}{2\cdot 3}\right).$$

For $\pi(x|\theta_1)$, we have

$$\pi(x|\theta_1) = \int \pi(x|\theta)\,\pi(\theta|\theta_1)\,d\theta,$$

that is, $\pi(x|\theta_1)$ is the marginal distribution of $\pi(x|\theta)$ and $\pi(\theta|\theta_1)$. By (3.2), we have

$$\pi(x|\theta_1) \sim N(\theta_1,2).$$

Hence,

$$\pi(x|\theta_1) = \frac{1}{\sqrt{2\pi}\sqrt{2}} \exp\left[-\frac{(x-\theta_1)^2}{2\cdot 2}\right].$$

For $\pi(x|\theta,\theta_1)$, we have

$$\pi(x|\theta,\theta_1) = \pi(x|\theta) \sim N(\theta,1).$$

Thus,

$$\pi(x|\theta,\theta_1) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(x-\theta)^2}{2}\right].$$

For $\pi(\theta, \theta_1|x)$, we have

$$\begin{aligned}\pi(\theta, \theta_1|x) &= \frac{\pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1)}{\pi(x)} \\ &= \frac{\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{(x-\theta)^2}{2}\right]\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{(\theta-\theta_1)^2}{2}\right]\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{\theta_1^2}{2}\right]}{\frac{1}{\sqrt{2\pi}\sqrt{3}}\exp\left(-\frac{x^2}{2\cdot 3}\right)} \\ &= \frac{\sqrt{3}}{2\pi}\exp\left\{-\frac{1}{2}\left[(x-\theta)^2+(\theta-\theta_1)^2+\theta_1^2-\frac{x^2}{3}\right]\right\} \\ &\propto \exp\left\{-\frac{1}{2}\left[(x-\theta)^2+(\theta-\theta_1)^2+\theta_1^2\right]\right\} \\ &\propto \exp\left\{-\frac{1}{2}\left[2\theta^2+2\theta_1^2-2\theta\theta_1-2x\theta\right]\right\} \\ &= \exp\left\{-\left[\theta^2+\theta_1^2-\theta\theta_1-x\theta\right]\right\}. \end{aligned} \tag{3.3}$$

It remains to show that $\pi(\theta, \theta_1|x)$ is a two-dimensional normal distribution $N_2(\mu_\theta, \mu_{\theta_1}, \sigma_\theta, \sigma_{\theta_1}, \rho)$ with appropriate parameter values. Let

$$c^2 = 2\left(1-\rho^2\right), \tag{3.4}$$

where

$$c = \sqrt{2(1-\rho^2)} > 0.$$

We have

$$\begin{aligned}\pi(\theta, \theta_1|x) &\propto \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{\theta-\mu_\theta}{\sigma_\theta}\right)^2+\left(\frac{\theta_1-\mu_{\theta_1}}{\sigma_{\theta_1}}\right)^2-2\rho\frac{\theta-\mu_\theta}{\sigma_\theta}\frac{\theta_1-\mu_{\theta_1}}{\sigma_{\theta_1}}\right]\right\} \\ &= \exp\left\{-\frac{1}{c^2}\left[\left(\frac{\theta-\mu_\theta}{\sigma_\theta}\right)^2+\left(\frac{\theta_1-\mu_{\theta_1}}{\sigma_{\theta_1}}\right)^2-2\rho\frac{\theta-\mu_\theta}{\sigma_\theta}\frac{\theta_1-\mu_{\theta_1}}{\sigma_{\theta_1}}\right]\right\} \\ &= \exp\left\{-\left[\left(\frac{\theta-\mu_\theta}{c\sigma_\theta}\right)^2+\left(\frac{\theta_1-\mu_{\theta_1}}{c\sigma_{\theta_1}}\right)^2-2\rho\frac{\theta-\mu_\theta}{c\sigma_\theta}\frac{\theta_1-\mu_{\theta_1}}{c\sigma_{\theta_1}}\right]\right\} \\ &= \exp\left\{-\left[\frac{\theta^2-2\mu_\theta\theta+\mu_\theta^2}{c^2\sigma_\theta^2}+\frac{\theta_1^2-2\mu_{\theta_1}\theta_1+\mu_{\theta_1}^2}{c^2\sigma_{\theta_1}^2}-\frac{2\rho}{c^2\sigma_\theta\sigma_{\theta_1}}\left(\theta\theta_1-\mu_{\theta_1}\theta-\mu_\theta\theta_1+\mu_\theta\mu_{\theta_1}\right)\right]\right\} \\ &\propto \exp\left\{-\left[\frac{\theta^2}{c^2\sigma_\theta^2}+\frac{\theta_1^2}{c^2\sigma_{\theta_1}^2}-\frac{2\rho}{c^2\sigma_\theta\sigma_{\theta_1}}\theta\theta_1+\left(\frac{-2\mu_\theta}{c^2\sigma_\theta^2}+\frac{2\rho\mu_{\theta_1}}{c^2\sigma_\theta\sigma_{\theta_1}}\right)\theta+\left(\frac{-2\mu_{\theta_1}}{c^2\sigma_{\theta_1}^2}+\frac{2\rho\mu_\theta}{c^2\sigma_\theta\sigma_{\theta_1}}\right)\theta_1\right]\right\} \end{aligned} \tag{3.5}$$

Comparing (3.3) and (3.5), we find that

$$\begin{aligned}&\theta^2+\theta_1^2-\theta\theta_1-x\theta \\ &= \frac{\theta^2}{c^2\sigma_\theta^2}+\frac{\theta_1^2}{c^2\sigma_{\theta_1}^2}-\frac{2\rho}{c^2\sigma_\theta\sigma_{\theta_1}}\theta\theta_1+\left(\frac{-2\mu_\theta}{c^2\sigma_\theta^2}+\frac{2\rho\mu_{\theta_1}}{c^2\sigma_\theta\sigma_{\theta_1}}\right)\theta+\left(\frac{-2\mu_{\theta_1}}{c^2\sigma_{\theta_1}^2}+\frac{2\rho\mu_\theta}{c^2\sigma_\theta\sigma_{\theta_1}}\right)\theta_1.\end{aligned}$$

Comparing the coefficients of θ^2, θ_1^2, $\theta\theta_1$, θ, and θ_1, we obtain

$$\left\{\begin{array}{ccc}\frac{1}{c^2\sigma_\theta^2} & = & 1, \\ \frac{1}{c^2\sigma_{\theta_1}^2} & = & 1, \\ -\frac{2\rho}{c^2\sigma_\theta\sigma_{\theta_1}} & = & -1, \\ \frac{-2\mu_\theta}{c^2\sigma_\theta^2}+\frac{2\rho\mu_{\theta_1}}{c^2\sigma_\theta\sigma_{\theta_1}} & = & -x, \\ \frac{-2\mu_{\theta_1}}{c^2\sigma_{\theta_1}^2}+\frac{2\rho\mu_\theta}{c^2\sigma_\theta\sigma_{\theta_1}} & = & 0.\end{array}\right. \tag{3.6}$$

By noting (3.4) and solving the first three equations of (3.6), we easily obtain

$$\sigma_\theta = \sqrt{\frac{2}{3}},\ \sigma_{\theta_1} = \sqrt{\frac{2}{3}},\ \rho = \frac{1}{2}.$$

Substituting the values of σ_θ, σ_{θ_1}, and ρ into the last two equations of (3.6), we easily obtain

$$\mu_\theta = \frac{2}{3}x,\ \mu_{\theta_1} = \frac{1}{3}x.$$

Consequently,

$$\pi(\theta,\theta_1|x) \sim N_2\left(\mu_\theta = \frac{2}{3}x, \mu_{\theta_1} = \frac{1}{3}x, \sigma_\theta = \sqrt{\frac{2}{3}}, \sigma_{\theta_1} = \sqrt{\frac{2}{3}}, \rho = \frac{1}{2}\right).$$

For $\pi(\theta,\theta_1)$, we have

$$\begin{aligned}
\pi(\theta,\theta_1) &= \pi(\theta|\theta_1)\,\pi(\theta_1) \\
&= \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{(\theta-\theta_1)^2}{2}\right]\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{\theta_1^2}{2}\right] \\
&= \frac{1}{2\pi}\exp\left\{-\frac{1}{2}\left[(\theta-\theta_1)^2+\theta_1^2\right]\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left[\theta^2+2\theta_1^2-2\theta\theta_1\right]\right\} \\
&= \exp\left\{-\left[\frac{1}{2}\theta^2+\theta_1^2-\theta\theta_1\right]\right\} \\
&= \exp\left\{-\left[\left(\frac{\theta-0}{\sqrt{2}}\right)^2+\left(\frac{\theta_1-0}{1}\right)^2-\theta\theta_1\right]\right\}.
\end{aligned} \tag{3.7}$$

It remains to show that $\pi(\theta,\theta_1)$ is a two-dimensional normal distribution $N_2(\mu_\theta,\mu_{\theta_1},\sigma_\theta,\sigma_{\theta_1},\rho)$ with appropriate parameter values. Let c^2 be given by (3.4). We have

$$\begin{aligned}
\pi(\theta,\theta_1) &\propto \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{\theta-\mu_\theta}{\sigma_\theta}\right)^2+\left(\frac{\theta_1-\mu_{\theta_1}}{\sigma_{\theta_1}}\right)^2-2\rho\frac{\theta-\mu_\theta}{\sigma_\theta}\frac{\theta_1-\mu_{\theta_1}}{\sigma_{\theta_1}}\right]\right\} \\
&= \exp\left\{-\frac{1}{c^2}\left[\left(\frac{\theta-\mu_\theta}{\sigma_\theta}\right)^2+\left(\frac{\theta_1-\mu_{\theta_1}}{\sigma_{\theta_1}}\right)^2-2\rho\frac{\theta-\mu_\theta}{\sigma_\theta}\frac{\theta_1-\mu_{\theta_1}}{\sigma_{\theta_1}}\right]\right\} \\
&= \exp\left\{-\left[\left(\frac{\theta-\mu_\theta}{c\sigma_\theta}\right)^2+\left(\frac{\theta_1-\mu_{\theta_1}}{c\sigma_{\theta_1}}\right)^2-2\rho\frac{\theta-\mu_\theta}{c\sigma_\theta}\frac{\theta_1-\mu_{\theta_1}}{c\sigma_{\theta_1}}\right]\right\}.
\end{aligned} \tag{3.8}$$

Comparing (3.7) and (3.8), we find that

$$\begin{aligned}
&\left(\frac{\theta-0}{\sqrt{2}}\right)^2+\left(\frac{\theta_1-0}{1}\right)^2-\theta\theta_1 \\
&= \left(\frac{\theta-\mu_\theta}{c\sigma_\theta}\right)^2+\left(\frac{\theta_1-\mu_{\theta_1}}{c\sigma_{\theta_1}}\right)^2-2\rho\frac{\theta-\mu_\theta}{c\sigma_\theta}\frac{\theta_1-\mu_{\theta_1}}{c\sigma_{\theta_1}}.
\end{aligned}$$

Comparing the corresponding terms, we obtain

$$\left\{\begin{array}{ccc}
\mu_\theta &=& 0, \\
\mu_{\theta_1} &=& 0, \\
c\sigma_\theta &=& \sqrt{2}, \\
c\sigma_{\theta_1} &=& 1, \\
\frac{-2\rho}{c^2\sigma_\theta\sigma_{\theta_1}} &=& -1.
\end{array}\right. \tag{3.9}$$

By noting (3.4) and solving the last three equations of (3.9), we easily obtain

$$\sigma_\theta = \sqrt{2},\ \sigma_{\theta_1} = 1,\ \rho = \frac{1}{\sqrt{2}}.$$

Consequently,

$$\pi\left(\theta, \theta_1\right) \sim N_2\left(\mu_\theta = 0, \mu_{\theta_1} = 0, \sigma_\theta = \sqrt{2}, \sigma_{\theta_1} = 1, \rho = \frac{1}{\sqrt{2}}\right).$$

Now we calculate the 5 densities related to θ. For $\pi\left(\theta\right)$, we have

$$\pi\left(\theta\right) = \int \pi\left(\theta|\theta_1\right)\pi\left(\theta_1\right)d\theta_1.$$

That is, $\pi\left(\theta\right)$ is the marginal distribution of $\pi\left(\theta|\theta_1\right)$ and $\pi\left(\theta_1\right)$. From (3.1) and (3.2), we can easily obtain

$$\pi\left(\theta\right) \sim N\left(0, 2\right).$$

Therefore,

$$\pi\left(\theta\right) = \frac{1}{\sqrt{2\pi}\sqrt{2}}\exp\left(-\frac{\theta^2}{2\cdot 2}\right).$$

For $\pi\left(\theta|x\right)$, we have

$$\begin{aligned}
\pi\left(\theta|x\right) &\propto \int \pi\left(x|\theta\right)\pi\left(\theta|\theta_1\right)\pi\left(\theta_1\right)d\theta_1 \\
&\propto \int \exp\left[-\frac{\left(x-\theta\right)^2}{2}\right]\exp\left[-\frac{\left(\theta-\theta_1\right)^2}{2}\right]\exp\left[-\frac{\theta_1^2}{2}\right]d\theta_1 \\
&= \exp\left[-\frac{\left(x-\theta\right)^2}{2}\right]\int \exp\left\{-\frac{1}{2}\left[\left(\theta-\theta_1\right)^2+\theta_1^2\right]\right\}d\theta_1 \\
&\equiv \exp\left[-\frac{\left(x-\theta\right)^2}{2}\right]\cdot I_1,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \int \exp\left\{-\frac{1}{2}\left[\left(\theta-\theta_1\right)^2+\theta_1^2\right]\right\}d\theta_1 = \int \exp\left\{-\frac{1}{2}\left[2\theta_1^2-2\theta\theta_1+\theta^2\right]\right\}d\theta_1 \\
&= \int \exp\left\{-\left[\theta_1^2-\theta\theta_1+\frac{1}{2}\theta^2\right]\right\}d\theta_1 = \int \exp\left\{-\left[\left(\theta_1-\frac{1}{2}\theta\right)^2+\frac{1}{4}\theta^2\right]\right\}d\theta_1 \\
&= \int \exp\left\{-\left(\theta_1-\frac{1}{2}\theta\right)^2-\frac{1}{4}\theta^2\right\}d\theta_1 = \exp\left(-\frac{1}{4}\theta^2\right)\cdot\int \exp\left\{-\left(\theta_1-\frac{1}{2}\theta\right)^2\right\}d\theta_1.
\end{aligned}$$

Note that the probability density function (pdf) of a normal distribution integrates to 1, that is,

$$\begin{aligned}
&\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\tilde{\sigma}}\exp\left[-\frac{\left(x-\mu\right)^2}{2\tilde{\sigma}^2}\right]dx = 1 \\
\Leftrightarrow &\int_{-\infty}^{\infty}\exp\left[-\frac{\left(x-\mu\right)^2}{2\tilde{\sigma}^2}\right]dx = \sqrt{2\pi}\tilde{\sigma}. \qquad (3.10)
\end{aligned}$$

By (3.10), we have

$$I_1 = \exp\left(-\frac{1}{4}\theta^2\right)\cdot\sqrt{2\pi}\frac{1}{\sqrt{2}} \propto \exp\left(-\frac{1}{4}\theta^2\right).$$

Therefore,

$$\begin{aligned}\pi\left(\theta|x\right) &\propto \exp\left[-\frac{(x-\theta)^2}{2}\right]\exp\left(-\frac{1}{4}\theta^2\right)=\exp\left\{-\frac{1}{2}\left[(x-\theta)^2+\frac{1}{2}\theta^2\right]\right\}\\ &\propto \exp\left\{-\frac{1}{2}\left[\frac{3}{2}\theta^2-2x\theta\right]\right\}=\exp\left\{-\frac{3}{4}\left[\theta^2-\frac{4}{3}x\theta\right]\right\}=\exp\left\{-\frac{3}{4}\left[\left(\theta-\frac{2}{3}x\right)^2-\frac{4}{9}x^2\right]\right\}\\ &\propto \exp\left\{-\frac{3}{4}\left(\theta-\frac{2}{3}x\right)^2\right\}=\exp\left\{-\frac{1}{2\cdot\frac{2}{3}}\left(\theta-\frac{2}{3}x\right)^2\right\}\\ &\sim N\left(\frac{2}{3}x,\frac{2}{3}\right).\end{aligned}$$

Hence,

$$\pi\left(\theta|x\right)=\frac{1}{\sqrt{2\pi}\sqrt{\frac{2}{3}}}\exp\left\{-\frac{1}{2\cdot\frac{2}{3}}\left(\theta-\frac{2}{3}x\right)^2\right\}.$$

For $\pi\left(\theta|x,\theta_1\right)$, we have

$$\begin{aligned}\pi\left(\theta|x,\theta_1\right) &\propto \pi\left(x|\theta\right)\pi\left(\theta|\theta_1\right)\\ &\propto \exp\left[-\frac{(x-\theta)^2}{2}\right]\exp\left[-\frac{(\theta-\theta_1)^2}{2}\right]=\exp\left\{-\frac{1}{2}\left[(x-\theta)^2+(\theta-\theta_1)^2\right]\right\}\\ &= \exp\left\{-\frac{1}{2}\left[2\theta^2-2x\theta-2\theta_1\theta+x^2+\theta_1^2\right]\right\}\\ &\propto \exp\left\{-\left[\theta^2-\left(x+\theta_1\right)\theta\right]\right\}=\exp\left\{-\left[\left(\theta-\frac{x+\theta_1}{2}\right)^2-\frac{(x+\theta_1)^2}{4}\right]\right\}\\ &\propto \exp\left\{-\left(\theta-\frac{x+\theta_1}{2}\right)^2\right\}\\ &\sim N\left(\frac{x+\theta_1}{2},\frac{1}{2}\right).\end{aligned}$$

Hence,

$$\pi\left(\theta|x,\theta_1\right)=\frac{1}{\sqrt{2\pi}\frac{1}{\sqrt{2}}}\exp\left\{-\frac{1}{2\cdot\frac{1}{2}}\left(\theta-\frac{x+\theta_1}{2}\right)^2\right\}.$$

For $\pi\left(x,\theta_1|\theta\right)$, we have

$$\begin{aligned}\pi\left(x,\theta_1|\theta\right) &\propto \pi\left(x|\theta\right)\pi\left(\theta|\theta_1\right)\pi\left(\theta_1\right)\\ &\propto \exp\left\{-\frac{1}{2}\left[(x-\theta)^2+(\theta-\theta_1)^2+\theta_1^2\right]\right\}\\ &\propto \exp\left\{-\frac{1}{2}\left[(x-\theta)^2+2\left(\theta_1^2-\theta\theta_1\right)\right]\right\}\\ &\propto \exp\left\{-\frac{1}{2}\left[(x-\theta)^2+2\left(\theta_1-\frac{\theta}{2}\right)^2\right]\right\}\\ &= \exp\left\{-\left[\left(\frac{x-\theta}{\sqrt{2}}\right)^2+\left(\frac{\theta_1-\frac{\theta}{2}}{1}\right)^2\right]\right\}.\end{aligned}\tag{3.11}$$

It remains to show that $\pi(x,\theta_1|\theta)$ is a two-dimensional normal distribution $N_2(\mu_x,\mu_{\theta_1},\sigma_x,\sigma_{\theta_1},\rho)$ with appropriate parameter values. Let c^2 be given by (3.4). We have

$$\begin{aligned}\pi(x,\theta_1|\theta) &\propto \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2+\left(\frac{\theta_1-\mu_{\theta_1}}{\sigma_{\theta_1}}\right)^2-2\rho\frac{x-\mu_x}{\sigma_x}\frac{\theta_1-\mu_{\theta_1}}{\sigma_{\theta_1}}\right]\right\}\\ &= \exp\left\{-\frac{1}{c^2}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2+\left(\frac{\theta_1-\mu_{\theta_1}}{\sigma_{\theta_1}}\right)^2-2\rho\frac{x-\mu_x}{\sigma_x}\frac{\theta_1-\mu_{\theta_1}}{\sigma_{\theta_1}}\right]\right\}\\ &= \exp\left\{-\left[\left(\frac{x-\mu_x}{c\sigma_x}\right)^2+\left(\frac{\theta_1-\mu_{\theta_1}}{c\sigma_{\theta_1}}\right)^2-2\rho\frac{x-\mu_x}{c\sigma_x}\frac{\theta_1-\mu_{\theta_1}}{c\sigma_{\theta_1}}\right]\right\}. \qquad (3.12)\end{aligned}$$

Comparing (3.11) and (3.12), we find that

$$\left(\frac{x-\theta}{\sqrt{2}}\right)^2+\left(\frac{\theta_1-\frac{\theta}{2}}{1}\right)^2=\left(\frac{x-\mu_x}{c\sigma_x}\right)^2+\left(\frac{\theta_1-\mu_{\theta_1}}{c\sigma_{\theta_1}}\right)^2-2\rho\frac{x-\mu_x}{c\sigma_x}\frac{\theta_1-\mu_{\theta_1}}{c\sigma_{\theta_1}}.$$

Comparing the corresponding terms, we obtain

$$\mu_x=\theta,\ \mu_{\theta_1}=\frac{\theta}{2}.$$

Moreover,

$$\begin{cases} c\sigma_x &= \sqrt{2},\\ c\sigma_{\theta_1} &= 1,\\ \frac{-2\rho}{c^2\sigma_x\sigma_{\theta_1}} &= 0.\end{cases}$$

By noting (3.4) and solving the above three equations, we easily obtain

$$\sigma_x=1,\ \sigma_{\theta_1}=\frac{1}{\sqrt{2}},\ \rho=0.$$

Consequently,

$$\pi(x,\theta_1|\theta)\sim N_2\left(\mu_x=\theta,\mu_{\theta_1}=\frac{\theta}{2},\sigma_x=1,\sigma_{\theta_1}=\frac{1}{\sqrt{2}},\rho=0\right).$$

For $\pi(x,\theta_1)$, we have

$$\begin{aligned}\pi(x,\theta_1) &\propto \pi(x|\theta_1)\,\pi(\theta_1)\\ &\propto \exp\left\{-\frac{1}{4}(x-\theta_1)^2\right\}\exp\left\{-\frac{1}{2}\theta_1^2\right\}=\exp\left\{-\left[\frac{1}{4}(x-\theta_1)^2+\frac{1}{2}\theta_1^2\right]\right\}\\ &= \exp\left\{-\left[\frac{1}{4}\left(x^2-2x\theta_1+\theta_1^2\right)+\frac{1}{2}\theta_1^2\right]\right\}=\exp\left\{-\left[\frac{1}{4}x^2+\frac{3}{4}\theta_1^2-\frac{1}{2}x\theta_1\right]\right\}\\ &= \exp\left\{-\left[\left(\frac{x-0}{2}\right)^2+\left(\frac{\theta_1-0}{2/\sqrt{3}}\right)^2-\frac{1}{2}x\theta_1\right]\right\}. \qquad (3.13)\end{aligned}$$

It remains to show that $\pi(x,\theta_1)$ is a two-dimensional normal distribution $N_2(\mu_x,\mu_{\theta_1},\sigma_x,\sigma_{\theta_1},\rho)$ with appropriate parameter values. Let c^2 be given by (3.4). We have

$$\begin{aligned}\pi(x,\theta_1) &\propto \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2+\left(\frac{\theta_1-\mu_{\theta_1}}{\sigma_{\theta_1}}\right)^2-2\rho\frac{x-\mu_x}{\sigma_x}\frac{\theta_1-\mu_{\theta_1}}{\sigma_{\theta_1}}\right]\right\}\\ &= \exp\left\{-\frac{1}{c^2}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2+\left(\frac{\theta_1-\mu_{\theta_1}}{\sigma_{\theta_1}}\right)^2-2\rho\frac{x-\mu_x}{\sigma_x}\frac{\theta_1-\mu_{\theta_1}}{\sigma_{\theta_1}}\right]\right\}\\ &= \exp\left\{-\left[\left(\frac{x-\mu_x}{c\sigma_x}\right)^2+\left(\frac{\theta_1-\mu_{\theta_1}}{c\sigma_{\theta_1}}\right)^2-2\rho\frac{x-\mu_x}{c\sigma_x}\frac{\theta_1-\mu_{\theta_1}}{c\sigma_{\theta_1}}\right]\right\}. \qquad (3.14)\end{aligned}$$

Comparing (3.13) and (3.14), we find that

$$\left(\frac{x-0}{2}\right)^2+\left(\frac{\theta_1-0}{2/\sqrt{3}}\right)^2-\frac{1}{2}x\theta_1=\left(\frac{x-\mu_x}{c\sigma_x}\right)^2+\left(\frac{\theta_1-\mu_{\theta_1}}{c\sigma_{\theta_1}}\right)^2-2\rho\frac{x-\mu_x}{c\sigma_x}\frac{\theta_1-\mu_{\theta_1}}{c\sigma_{\theta_1}}.$$

Comparing the corresponding terms, we obtain

$$\mu_x=0,\ \mu_{\theta_1}=0.$$

Moreover,

$$\begin{cases} c\sigma_x & = & 2, \\ c\sigma_{\theta_1} & = & \frac{2}{\sqrt{3}}, \\ \frac{-2\rho}{c^2\sigma_x\sigma_{\theta_1}} & = & -\frac{1}{2}. \end{cases}$$

By noting (3.4) and solving the above three equations, we easily obtain

$$\sigma_x=\sqrt{3},\ \sigma_{\theta_1}=1,\ \rho=\frac{1}{\sqrt{3}}.$$

Consequently,

$$\pi\left(x,\theta_1\right)\sim N_2\left(\mu_x=0,\mu_{\theta_1}=0,\sigma_x=\sqrt{3},\sigma_{\theta_1}=1,\rho=\frac{1}{\sqrt{3}}\right).$$

Now we calculate the 5 densities related to θ_1. For $\pi\left(\theta_1|x\right)$, we have

$$\begin{aligned}\pi\left(\theta_1|x\right)&=\frac{\pi\left(x,\theta_1\right)}{\pi\left(x\right)}\\&\propto\frac{\exp\left\{-\left[\frac{1}{4}x^2+\frac{3}{4}\theta_1^2-\frac{1}{2}x\theta_1\right]\right\}}{\exp\left(-\frac{x^2}{6}\right)}\\&\propto\exp\left\{-\left[\frac{3}{4}\theta_1^2-\frac{1}{2}x\theta_1\right]\right\}=\exp\left\{-\frac{3}{4}\left[\theta_1^2-\frac{2}{3}x\theta_1\right]\right\}\\&\propto\exp\left\{-\frac{1}{2\cdot\frac{2}{3}}\left[\theta_1-\frac{1}{3}x\right]^2\right\}\\&\sim N\left(\frac{1}{3}x,\frac{2}{3}\right).\end{aligned}$$

Hence,

$$\pi\left(\theta_1|x\right)=\frac{1}{\sqrt{2\pi}\sqrt{\frac{2}{3}}}\exp\left\{-\frac{1}{2\cdot\frac{2}{3}}\left[\theta_1-\frac{1}{3}x\right]^2\right\}.$$

For $\pi\left(\theta_1|\theta\right)$, we have

$$\begin{aligned}\pi\left(\theta_1|\theta\right)&\propto\pi\left(\theta|\theta_1\right)\pi\left(\theta_1\right)\\&\propto\exp\left[-\frac{\left(\theta-\theta_1\right)^2}{2}\right]\exp\left[-\frac{\theta_1^2}{2}\right]=\exp\left\{-\frac{1}{2}\left[\left(\theta-\theta_1\right)^2+\theta_1^2\right]\right\}\\&=\exp\left\{-\frac{1}{2}\left[2\theta_1^2-2\theta\theta_1\right]\right\}=\exp\left\{-\left[\theta_1^2-\theta\theta_1\right]\right\}\\&\propto\exp\left\{-\left(\theta_1-\frac{\theta}{2}\right)^2\right\}\\&\sim N\left(\frac{\theta}{2},\frac{1}{2}\right).\end{aligned}$$

Another method to determine the distribution of $\pi(\theta_1|\theta)$ is by utilizing the Bayesian tool (3.2). We have

$$\begin{cases} \pi(\theta|\theta_1) \sim N\left(\theta_1, \frac{1}{1}\right), \\ \pi(\theta_1) \sim N\left(0, \frac{1}{1}\right). \end{cases} \Longrightarrow \theta_1|\theta \sim N\left(\frac{1\cdot 0 + 1\cdot\theta}{1+1}, \frac{1}{1+1}\right) = N\left(\frac{\theta}{2}, \frac{1}{2}\right).$$

Hence,

$$\pi(\theta_1|\theta) = \frac{1}{\sqrt{2\pi}\frac{1}{\sqrt{2}}} \exp\left\{-\frac{1}{2\cdot\frac{1}{2}}\left[\theta_1 - \frac{\theta}{2}\right]^2\right\}.$$

For $\pi(\theta_1|x,\theta)$, we have

$$\pi(\theta_1|x,\theta) = \pi(\theta_1|\theta) = \frac{1}{\sqrt{2\pi}\frac{1}{\sqrt{2}}} \exp\left\{-\frac{1}{2\cdot\frac{1}{2}}\left[\theta_1 - \frac{\theta}{2}\right]^2\right\} \sim N\left(\frac{\theta}{2}, \frac{1}{2}\right).$$

For $\pi(x,\theta|\theta_1)$, we have

$$\begin{aligned} \pi(x,\theta|\theta_1) &= \pi(x|\theta)\,\pi(\theta|\theta_1) \\ &\propto \exp\left[-\frac{(x-\theta)^2}{2}\right]\exp\left[-\frac{(\theta-\theta_1)^2}{2}\right] = \exp\left\{-\frac{1}{2}\left[(x-\theta)^2 + (\theta-\theta_1)^2\right]\right\} \\ &= \exp\left\{-\frac{1}{2}\left[((x-\theta_1)-(\theta-\theta_1))^2 + (\theta-\theta_1)^2\right]\right\} \\ &= \exp\left\{-\frac{1}{2}\left[(x-\theta_1)^2 + 2(\theta-\theta_1)^2 - 2(x-\theta_1)(\theta-\theta_1)\right]\right\} \\ &= \exp\left\{-\left[\left(\frac{x-\theta_1}{\sqrt{2}}\right)^2 + \left(\frac{\theta-\theta_1}{1}\right)^2 - (x-\theta_1)(\theta-\theta_1)\right]\right\}. \end{aligned} \tag{3.15}$$

It remains to show that $\pi(x,\theta|\theta_1)$ is a two-dimensional normal distribution $N_2(\mu_x,\mu_\theta,\sigma_x,\sigma_\theta,\rho)$ with appropriate parameter values. Let c^2 be given by (3.4). We have

$$\begin{aligned} \pi(x,\theta|\theta_1) &\propto \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{\theta-\mu_\theta}{\sigma_\theta}\right)^2 - 2\rho\frac{x-\mu_x}{\sigma_x}\frac{\theta-\mu_\theta}{\sigma_\theta}\right]\right\} \\ &= \exp\left\{-\frac{1}{c^2}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{\theta-\mu_\theta}{\sigma_\theta}\right)^2 - 2\rho\frac{x-\mu_x}{\sigma_x}\frac{\theta-\mu_\theta}{\sigma_\theta}\right]\right\} \\ &= \exp\left\{-\left[\left(\frac{x-\mu_x}{c\sigma_x}\right)^2 + \left(\frac{\theta-\mu_\theta}{c\sigma_\theta}\right)^2 - 2\rho\frac{x-\mu_x}{c\sigma_x}\frac{\theta-\mu_\theta}{c\sigma_\theta}\right]\right\}. \end{aligned} \tag{3.16}$$

Comparing (3.15) and (3.16), we find that

$$\left(\frac{x-\theta_1}{\sqrt{2}}\right)^2 + \left(\frac{\theta-\theta_1}{1}\right)^2 - (x-\theta_1)(\theta-\theta_1) = \left(\frac{x-\mu_x}{c\sigma_x}\right)^2 + \left(\frac{\theta-\mu_\theta}{c\sigma_\theta}\right)^2 - 2\rho\frac{x-\mu_x}{c\sigma_x}\frac{\theta-\mu_\theta}{c\sigma_\theta}.$$

Comparing the corresponding terms, we obtain

$$\mu_x = \theta_1,\ \mu_\theta = \theta_1.$$

Moreover,

$$\begin{cases} c\sigma_x &= \sqrt{2}, \\ c\sigma_\theta &= 1, \\ \frac{-2\rho}{c^2\sigma_x\sigma_\theta} &= -1. \end{cases}$$

By noting (3.4) and solving the above three equations, we easily obtain

$$\sigma_x = \sqrt{2},\ \sigma_\theta = 1,\ \rho = \frac{1}{\sqrt{2}}.$$

Consequently,

$$\pi\left(x, \theta|\theta_1\right) \sim N_2\left(\mu_x = \theta_1, \mu_\theta = \theta_1, \sigma_x = \sqrt{2}, \sigma_\theta = 1, \rho = \frac{1}{\sqrt{2}}\right).$$

For $\pi\left(x, \theta\right)$, we have

$$\begin{aligned}
\pi\left(x, \theta\right) &= \pi\left(\theta|x\right)\pi\left(x\right) \\
&= \frac{1}{\sqrt{2\pi}\sqrt{\frac{2}{3}}}\exp\left\{-\frac{1}{2\cdot\frac{2}{3}}\left(\theta - \frac{2}{3}x\right)^2\right\}\frac{1}{\sqrt{2\pi}\sqrt{3}}\exp\left(-\frac{x^2}{2\cdot 3}\right) \\
&= \frac{1}{2\pi\sqrt{2}}\exp\left[-\frac{3}{4}\left(\theta - \frac{2}{3}x\right)^2 - \frac{x^2}{6}\right] \\
&\propto \exp\left\{-\left[\frac{3}{4}\left(\theta - \frac{2}{3}x\right)^2 + \frac{x^2}{6}\right]\right\} = \exp\left\{-\left[\frac{x^2}{2} + \frac{\theta^2}{\frac{4}{3}} - x\theta\right]\right\} \\
&= \exp\left\{-\left[\left(\frac{x-0}{\sqrt{2}}\right)^2 + \left(\frac{\theta - 0}{\frac{2}{\sqrt{3}}}\right)^2 - x\theta\right]\right\}. \qquad (3.17)
\end{aligned}$$

It remains to show that $\pi\left(x, \theta\right)$ is a two-dimensional normal distribution $N_2\left(\mu_x, \mu_\theta, \sigma_x, \sigma_\theta, \rho\right)$ with appropriate parameter values. Let c^2 be given by (3.4). We have

$$\begin{aligned}
\pi\left(x, \theta\right) &\propto \exp\left\{-\frac{1}{2\left(1-\rho^2\right)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{\theta-\mu_\theta}{\sigma_\theta}\right)^2 - 2\rho\frac{x-\mu_x}{\sigma_x}\frac{\theta-\mu_\theta}{\sigma_\theta}\right]\right\} \\
&= \exp\left\{-\frac{1}{c^2}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{\theta-\mu_\theta}{\sigma_\theta}\right)^2 - 2\rho\frac{x-\mu_x}{\sigma_x}\frac{\theta-\mu_\theta}{\sigma_\theta}\right]\right\} \\
&= \exp\left\{-\left[\left(\frac{x-\mu_x}{c\sigma_x}\right)^2 + \left(\frac{\theta-\mu_\theta}{c\sigma_\theta}\right)^2 - 2\rho\frac{x-\mu_x}{c\sigma_x}\frac{\theta-\mu_\theta}{c\sigma_\theta}\right]\right\}. \qquad (3.18)
\end{aligned}$$

Comparing (3.17) and (3.18), we find that

$$\left(\frac{x-0}{\sqrt{2}}\right)^2 + \left(\frac{\theta-0}{\frac{2}{\sqrt{3}}}\right)^2 - x\theta = \left(\frac{x-\mu_x}{c\sigma_x}\right)^2 + \left(\frac{\theta-\mu_\theta}{c\sigma_\theta}\right)^2 - 2\rho\frac{x-\mu_x}{c\sigma_x}\frac{\theta-\mu_\theta}{c\sigma_\theta}.$$

Comparing the corresponding terms, we obtain

$$\mu_x = 0,\ \mu_\theta = 0.$$

Moreover,

$$\left\{\begin{array}{ccc} c\sigma_x & = & \sqrt{2}, \\ c\sigma_\theta & = & \frac{2}{\sqrt{3}}, \\ \frac{-2\rho}{c^2\sigma_x\sigma_\theta} & = & -1. \end{array}\right.$$

By noting (3.4) and solving the above three equations, we easily obtain

$$\sigma_x = \sqrt{3},\ \sigma_\theta = \sqrt{2},\ \rho = \sqrt{\frac{2}{3}}.$$

Consequently,

$$\pi(x,\theta) \sim N_2\left(\mu_x = 0, \mu_\theta = 0, \sigma_x = \sqrt{3}, \sigma_\theta = \sqrt{2}, \rho = \sqrt{\frac{2}{3}}\right).$$

Finally, for $\pi(x,\theta,\theta_1)$, we have

$$\begin{aligned}
\pi(x,\theta,\theta_1) &= \pi(x|\theta)\,\pi(\theta|\theta_1)\,\pi(\theta_1) \\
&= \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{(x-\theta)^2}{2}\right]\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{(\theta-\theta_1)^2}{2}\right]\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{\theta_1^2}{2}\right] \\
&= (2\pi)^{-\frac{3}{2}}\exp\left\{-\frac{1}{2}\left[(x-\theta)^2+(\theta-\theta_1)^2+\theta_1^2\right]\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left[(x-\theta)^2+(\theta-\theta_1)^2+\theta_1^2\right]\right\}. \qquad (3.19)
\end{aligned}$$

It remains to show that $\pi(x,\theta,\theta_1)$ is a three-dimensional normal distribution $N_3(\boldsymbol{\mu},\boldsymbol{\Sigma})$ with appropriate parameter values, where

$$\boldsymbol{\mu} = (\mu_x, \mu_\theta, \mu_{\theta_1})', \ \boldsymbol{\Sigma}^{-1} = A = (a_{ij})_{3\times 3}.$$

Let $\boldsymbol{x} = (x,\theta,\theta_1)'$. We have

$$\pi(x,\theta,\theta_1) \propto \exp\left\{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right\} = \exp\left\{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})'A(\boldsymbol{x}-\boldsymbol{\mu})\right\}. \qquad (3.20)$$

Comparing (3.19) and (3.20), we find that

$$\begin{aligned}
(\boldsymbol{x}-\boldsymbol{\mu})'A(\boldsymbol{x}-\boldsymbol{\mu}) &= (x-\theta)^2+(\theta-\theta_1)^2+\theta_1^2 \\
&= x^2+2\theta^2+2\theta_1^2-2x\theta-2\theta\theta_1 \\
&= (x,\theta,\theta_1)\begin{pmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{pmatrix}\begin{pmatrix}x\\ \theta\\ \theta_1\end{pmatrix}.
\end{aligned}$$

Comparing the corresponding terms, we obtain

$$\begin{aligned}
\boldsymbol{\mu} &= (0,0,0)', \\
\boldsymbol{\Sigma}^{-1} = A &= \begin{pmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{pmatrix}, \\
\boldsymbol{\Sigma} = A^{-1} &= \begin{pmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{pmatrix}^{-1} = \begin{pmatrix}3 & 2 & 1\\ 2 & 2 & 1\\ 1 & 1 & 1\end{pmatrix}.
\end{aligned}$$

4 Conclusions

There are 19 densities involved in the hierarchical Bayes model with two conditional levels, in which the 3 densities $\pi(x|\theta)$, $\pi(\theta|\theta_1)$, and $\pi(\theta_1)$ are known densities. Fig. 1 provides these 19 densities. Note that 4 of the 16 unknown densities are obviously represented by the 3 known densities. The remaining 12 unknown densities, their dependence densities, and the order of calculation are summarized in Table 1. After that, we have written the 16 unknown densities in terms of the 3 known densities in Theorem 2.1 which is very handy for practitioners and researchers interested in the hierarchical Bayes model with two conditional levels. Finally, we apply Theorem 2.1 to a hierarchical normal Bayes model with two conditional levels and obtain the functional forms of the 16 densities. Moreover, for the simple hierarchical normal Bayes model, we figure out the exact distributions of the 16 densities, which are one-, two-, or three-dimensional normal distributions. In other hierarchical Bayes models, one may not obtain analytical expressions of the densities, then one should be able to derive the densities numerically.

Acknowledgement

The authors gratefully acknowledge the constructive comments offered by the referees. Their comments improve the quality of the paper significantly. The research was supported by the Fundamental Research Funds for the Central Universities (CQDXWL-2012-004; 106112016CDJXY 100002), China Scholarship Council (201606055028), National Natural Science Foundation of China (11671060), and MOE project of Humanities and Social Sciences on the west and the border area (14XJC910001).

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Lindley DV. Introduction to probability and statistics from a Bayesian viewpoint. Part 2. Inference. Cambridge University Press: Cambridge; 1965.

[2] DeGroot M. Optimal statistical decisions. McGraw-Hill: New York; 1970.

[3] Zellner A. An introduction to Bayesian inference in econometrics. Wiley: New York; 1971.

[4] Berger JO. Statistical decision theory and Bayesian analysis. Springer: New York, 2nd edition; 1985.

[5] Box GE, Tiao GC. Bayesian inference in statistical analysis. Wiley: New York; 1992.

[6] Bernardo JM, Smith AFM. Bayesian theory. Wiley: New York; 1994.

[7] Chen MH, Shao QM, Ibrahim JG. Monte carlo methods in Bayesian computation. Springer: New York; 2000.

[8] Robert CP. The Bayesian choice: From decision-theoretic motivations to computational implementation. Springer: New York, 2nd paperback edition; 2007.

[9] Albert J. Bayesian computation with R (Use R!). Springer: New York, 2nd edition; 2009.

[10] Liang F, Liu C, Carroll RJ. Advanced Markov Chain Monte Carlo: Learning from Past Samples. Wiley: New York; 2010.

[11] Mao SS, Tang YC. Bayesian statistics. China Statistics Press: Beijing. 2nd edition; 2012.

[12] Gelman A, Carlin JB, Stern HS, Dunson DB, Vehtari A, Rubin DB. Bayesian Data Analysis. Chapman & Hall: London, 3rd edition; 2013.

[13] Robbins H. An empirical bayes approach to statistics. Proceedings of Third Berkeley Symposium on Mathematical Statistics and Probability. University of California Press. 1955;1.

[14] Robbins H. The empirical bayes approach to statistical decision problems. Annals of Mathematical Statistics. 1964;35:1-20.

[15] Deely JJ, Lindley DV. Bayes empirical bayes. Journal of the American Statistical Association. 1981;76:833-841.

[16] Morris C. Parametric empirical bayes inference: Theory and applications. Journal of the American Statistical Association. 1983;78:47–65.

[17] Robbins H. Some thoughts on empirical bayes estimation. Annals of Statistics. 1983;1:713-723.

[18] Maritz JS, Lwin T. Empirical Bayes methods. Chapman & Hall: London. 2nd edition; 1989.

[19] Carlin BP, Louis A. Bayes and empirical Bayes methods for data analysis. Chapman & Hall: London, 2nd edition; 2000.

[20] Carlin BP, Louis A. Empirical bayes: Past, present and future. Journal of the American Statistical Association. 2000;95:1286-1290.

[21] Lindley DV, Smith AFM. Bayes estimates for the linear model. Journal of the Royal Statistical Society, Series B. 1972;34:1-41.

[22] Good IJ. Some history of the hierarchical bayesian methodology. Bayesian Statistics 2. Bernardo JM, DeGroot MH, Lindley DV, Smith AFM (eds.). North-Holland: Amsterdam; 1980.

[23] Good IJ. Good thinking: The foundations of probability and its applications. University of Minnesota Press: Minneapolis; 1983.

[24] Hobert JP. Hierarchical models: A current computational perspective. Journal of the American Statistical Association; 2000;95:1312-1316.

[25] Mosteller F, Chalmers TC. Some progress and problems in meta-analysis of clinical trials. Statistical Science. 1992; 7:227-236.

[26] Guihenneuc-Jouyaux C, Richardson S, Lasserre V. Convergence assessment in latent variable models: Application to longitudinal modelling of a marker of hiv progression. Discretization and MCMC Convergence Assessment. Lecture Notes in Statistics. Robert CP (ed.). chap. 7, Springer-Verlag: New York. 1998;135:147-160.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://sciencedomain.org/review-history/20812