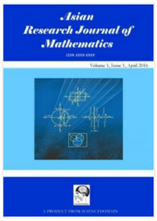

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On the Operation of Division by Zero in Bhaskara's Framework: Survey, Criticisms, Modifications and Justifications

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

Brahmagupta introduced zero into the genus of real numbers and discussed operations involving it. Of these the operation of division by zero has caused a lot of disputes among mathematicians. The ultimate aim of this work is to clarify and justify Bhaskara's arithmetic operation of division by zero and to justifiably demonstrate that the most significant case of division by zero, $1/0$, has the infinite number $(-1)!$ as its exact quotient.

Keywords: Unity; zero; infinity; euler's constant; minus one factorial; polytopic numbers.

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1 Introduction

To those acquainted with the history of mathematics it may seem as though all of the power of mathematicians must have been absorbed in foundational disputes over the quotient of division by zero [1], [2]. The subject is not found in any document or relic belonging to the Greek

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mathematicians of antiquity, so far as any remains or any record of them are preserved, coming down from ancient times, are, as regards this subject an absolute blank [1], [3].

Brahmagupta (born 598) appeared to be the first to attempt a definition for $a \div 0$. In his *Brahmasphula siddhanta*, he spoke of $a \div 0$ as being the fraction $a/0$. Read this great mathematician: 'Positive or negative numbers when divided by zero is a fraction with the zero as denominator'[4], [5]. He seemed to have believed that $a/0$ is irreducible.

Such was the condition of division by zero until about 1152 AD when the brilliant genius of Bhaskara seized upon an insight and boldly followed its lead until he developed an additional notion of zero divisor which has astonished and delighted the world. In his *Bijaganita*, we read him:

> A quantity divided by zero becomes a fraction the denominator of which is zero. This fraction is termed an infinite quantity. In this quantity consisting of that which has zero for its divisor, there is no alteration, though many may be inserted or extracted; as no change takes place in the infinite and immutable God when worlds are created or destroyed, though numerous orders of beings are absorbed or put forth. [6],[7], [8] [1].

Over the last three decades there has been a resurgence of writing on the concept of division by zero [9], indicating a debate concerning this key element of analysis. The chief goal of this paper is, therefore, to re-investigate division by zero, taking into account a resent research on the minus one factorial $(-1)!$, a number well known to be equal to $1/0$ [10], [11]. My wish throughout this work has been to enliven Bhaskara's notions of zero divisor, adverting to justifications of the identity $1/0 = (-1)!$ on which the readers may securely rest.

I wish to inform the readers that I myself am not worthy to judge the work itself. My desire is that it pleases those who have at heart the permitting of the operation of division by zero in mathematics. It is my greatest hope that this work furnishes insights that might be of importance to them or at least opens a way for new studies. We shall set forth a clear notion of division by zero, not in naked deformity, but softened and made as palatable as possible by associating it with applications that are undoubtedly true and undeniable. The reader will see in this work that $1/0$ is not undefined but actually a number.

The rest of this work is divided into four sections. Section 2 discusses the operation of division by zero as recorded in history. Section 3 is devoted to the discussion of how the number $(-1)!$ arises as the quotient of the fraction $1/0$ together with the manner in which it is related to the factorials of negative integers. Section 4 is concerned with the justification of the use, in analysis, of $(-1)!$ and $1/0$ together with Bhaskara's notions of division by zero.

2 Survey of Zero Arithmetic

Division by zero is treated in a facet of ancient Indian mathematics called Zero arithmetic (*sunya ganita*). In this branch relations and properties of zero are treated. Before we discuss the basics of this arithmetic, we present a rudimentary survey of zero arithmetic.

2.1 Indian mathematicians

Neglecting other trifling hints which may be found in preceding Indian writings and which prepared the subject for its birth, we may say that zero arithmetic really commenced with one of the greatest men of genius that the world has ever known, the Indian outstanding mathematician and astronomer of the 7th Century, Brahmagupta, and it would be difficult to find any other name which could confer higher honour on the subject, for he alone first formulated zero arithmetic by effectively making use of the concept of 0, which he referred to as *sunya* [4]. And the world is highly indebted to him for his valuable contributions to mathematics which has furnished the most excellent applications in every human activity.

Brahmagupta's *Brahmasphuta–siddhanta* (c. 628) is the first text to discuss the arithmetic of zero. There he gave the formal definition of zero:

> ... positive (*dhana*) and negative (*rna*), if they are equal, is zero (*kham*).

He then defined the operations of addition, subtraction, and multiplication involving zero:

> The sum of a negative and zero is a negative, of a positive and zero is positive and of two zeros, zero (*sunya*).... Negative subtracted from zero is positive, and positive from zero negative. Zero subtracted from negative is negative, from positive is positive, and from zero is zero .... The product of zero and a negative, of zero and a positive, of two zeros is zero.

To divide a finite number by zero is really a difficult task [9], [12], [13], [14]. Brahmagupta's experience with operations of addition, subtraction, multiplication and division of numbers led him to develop the intuitive belief that every operation must result in a number. By treating zero 0 as a number, he considered the division of any finite number by zero as a number also and called it *taccheda*. See his interesting remark:

> Positive or negative, divided by zero, is a fraction with zero for denominator. Zero divided by negative or positive is either zero or is expressed by a fraction with zero as numerator and the finite quantity as denominator. ... zero divided by zero is zero ....

The fundamental properties of 0 enunciated by Brahmagupta have inspired mankind ever since and, indeed, remain the ambition of all modern science. But Brahmagupta himself did not fully cover the arithmetic of zero as denominator.

His compatriots who came after him took up his enigmatic concept of zero denominator and attempted to clarify it further. Thus Mahavira wrote

> A number remains unchanged when it is divided by zero.

By this Mahavira was suggesting that the finite number acting as numerator of *taccheda* is not partitioned as the dividing number is absolute nothing. Thus *taccheda* has no determinate quotient.

Sripati (1039–1056) substituted the more elucidating term *khahara* for *taccheda*, for *khahara* signifies that which has zero (*kha*) as the divisor.

> A number when multiplied by zero becomes zero, and when divided by zero becomes *khahara*.

The condition of division by zero remained the same until about 1152 when the acute Bhaskara seized upon an insight and boldly followed its lead until he advanced a further notion of zero divisor which has amazed and delighted the world until this day. Bhaskara made important mathematical discoveries. Of these it is his doctrine on zero divisor that has had the greatest influence on subsequent mathematical discoveries. He set out to collect Brahmagupta's ideas of zero and to arrange them as a single self– contained whole. He entered into a detailed study of the properties of zero, became intimately acquainted with them, and thereby emerged as the most illustrious of all connoisseurs of zero arithmetic [6],[7], [1].

His *Lilavati* was his first systematic exposition of his ideas on zero. There we see his annotation on zero:

> When a number is added to zero the result is that number. The square, & c. of zero is zero. A number divided by zero has zero for its divisor. When a number is multiplied by zero the product is zero; ....

These statements are a review of the arithmetic of zero. The keynote is struck at once. Here we are shown, briefly but conclusively, the relations of zero. The genius introduced his subject in a manner least calculated to provoke his readers. He begins by acknowledging the already laid down properties of zero. He confirms and seals the doctrine of zero which was held by his fellow countrymen, that

$$0 + a = a, \qquad 0^2 = 0, \qquad 0^3 = 0, \qquad a/0 = a/0, \qquad a \cdot 0 = 0$$

It is worthy of note that his arithmetic of zero opens with a summary of Brahmagupta's properties of zero.

Having affirmed that $a \cdot 0 = 0$, the genius at once pointed out that in further operations on zero the multiple of zero $a \cdot 0$ in which 0 is a multiplier should be used. The great object of this novel idea is to restore the finite number a acting as the multiplicand of the multiple of 0:

> ...but in case any operation remains to be done, zero is merely conceived to be the multiplier, and when zero also becomes the divisor, the number is considered unchanged.

In this statement Bhaskara has set for many a philosophical problem. How entirely different is this remark from those of previous masters of zero? Every word in it calls for our most careful attention. It appears from this remark that Bhaskara was not content with Brahmagupta's rule on zero divided by zero. Brahmagupta had commented that the quotient of such division is zero. Bhaskara might have applied this rule in calculating the motions of the heavenly bodies, and found that it sometimes fails. In an attempt to resolve this situation, he discovered this novel law of impending operations on zero: Although $a \cdot 0 = 0$, the form $a \cdot 0$ should be used in upcoming operations on 0; and if in one of these operations we meet with the expression

$$\frac{a \cdot 0}{0}$$

we should consider $0/0 = 1$ so that

$$\frac{a \cdot 0}{0} = a.$$

Thus, for Bhaskara,

$$\frac{a \cdot 0}{0} \neq \frac{0}{0}$$

but

$$\frac{a \cdot 0}{0} = a.$$

The property $a \cdot 0 = 0$ was seen by him as that which holds only after the last operation on 0 is performed.

Bhaskara deserves to be esteemed in the world of mathematics for introducing this elegant law of impending operations on zero, for which India may be as well justly proud that such a law has not only been discovered by one of her own naturally endowed mathematicians and astronomers, but furnished with so worthy a dignity. In a short space, he unfolds his thinking on the eluding nature of the mathematical zero. By introducing this law Bhaskara had put 0 on equal footing with finite numbers, for if x is any finite number whatsoever, then

$$\frac{a \cdot x}{x} = a.$$

In closing his discussion on zero in his *Lilavati*, he presented some illustrative examples to show how his zero arithmetic is handled.

> What is the result of 5 added to zero? What are the square, square root, cube, and cube root of zero? What is the quotient if 10 is divided by zero? Required such a number that being multiplied by zero, and half of the result added to the product, this sum multiplied by three, and this last product divided by zero, the result shall be 63?

The answers given in the text are as follows: $5, 0, 0, 0, 0, 10/0, 14$. "This method of reckoning", says the genius, "is of great utility in calculating the motions of the heavenly bodies".

Now, we have to speak of his zero algebra in his *Bijaganita*. Here, he gave the first clear discussions of *khahara*. Again, he reviews the arithmetic of zero:

> In the addition of zero or subtraction of it, the quantity, positive or negative, remains the same. But, subtracted from zero, it is reversed. In multiplication and the rest of the operations of zero, the product is zero; and so it is in multiplication by zero; but a quantity, divided by zero, becomes a fraction the denominator of which is zero.

We see in the last statement that Bhaskara with certitude held to the doctrine of *khahara* being a fractional number. He in his examples stated that

$$\frac{3}{0} = \frac{3}{0}.$$

He then followed this up with one of the most famous quotations of all times:

> A quantity divided by zero becomes a fraction the denominator of which is zero. This fraction is termed an infinite quantity (*khahara*).

Bhaskara emphasized that whenever $a/0$ appears, it remains immutable in form; any finite number added to it or subtracted from it will not alter its value. The following stanza describes the nature of infinity (*khahara*).

> There is no change in *khahara* figure (infinite quantity) if something is added to or subtracted from the same. It is like there is no change in infinite Vishnu (Almighty) due to dissolution or creation of abounding beings.

2.2 European mathematicians

Hundred years past the death of Bhaskara, schools of analysis in Europe took up the concept of division by zero and they started developing a totally new concept that was both foreign to Bhaskara and Indian mathematicians after him. Prior to the advancement of calculus, there were apparently no books written on zero divisor in Europe, nor were there any controversies, for every mathematician knew that the sign 0 denotes absence of quantity. There was no discussion of zero and division by zero as a subject. Zero had never been referred to as infinitely small or vanishing quantity [15]. There was no need for debate given that this concept of zero divisor never crossed anyone's mind until the foundation of the calculus was questioned. Since then, there have been volumes written to try to explain the distinction between nothing and infinitely small. Why would it take volumes? The reason is that division by zero is inexplicable in light of its meaning of being division by absolute nothing or void.

In 1716 John Craig asserted that 0 cannot be an absolute nothing,

> ... for infinite number of absolute nothings cannot make 1, but by 0 is understood an infinitely small part, as in the calc. diff. dx is an infinitely small part of x, so that dx is as 0 to x. Not that dx is absolutely nothing, for it is divisible into an infinite number of parts each of which is d dx.

To make his ideas still plainer Craig continued:

> But then it may be inquired what is the quotient that arises from the division of 1 by absolute nothing. I say there is no quotient because there is no division. Therefore, it is a mistake to say the quotient is 1 or unity undivided which is demonstrably false, neither is the quotient= 0. For properly speaking there is no quotient and there it is an error to assign any.

The doctrine of zero divided by zero furnishing a finite quotient had been often described as absurd and warmly contested. Some have considered zero divided by zero, or the vanishing fraction, as it was first called in Europe as if zero were a quantity, to be meaningless or undefined probably because it was supposed that the fraction was indeterminate; it has however been objected to by others, and many ingenious arguments have been advanced on both sides, but without coming to a satisfactory conclusion. Two chief critics of this doctrine of vanishing fractions, Rolle and Berkeley, fancied that they had assured themselves of the absurdity of vanishing fractions from their analysis; from which they concluded that such fractions are meaningless.

A hint on the vehement opposition of Rolle to the doctrine of zero divided by zero furnishing a finite quotient is as follows:

> Such fractions as have both their numerator and denominator vanish, or equal to 0, at the same time, may be called vanishing fractions. We are not to conclude that such fractions are equal to nothing, or have no value; for that they have a certain determinate value, has been shown by the best mathematicians. The idea of such fractions as these, first originated in a severe contest among some French mathematicians, in which Varignon and Rolle were the chief opposite combatants; concerning the then new or differential calculus, of which the later gentleman was a strenuous opponent. Among other arguments against it, he proposed an example of drawing a tangent to certain curves at the point where the two parts cross each other; and as the fractional expression for the sub tangent, by that method, had both its numerator and denominator equal to 0 at the point proposed, Rolle considered it as an absurd expression, and as an argument against the method of solution itself. The seeming mystery however was soon explained, and first of all by John Bernoulli.

The doctrine of the validity of the vanishing fractions has however found an able advocate in the ingenious Euler who has given an explanatory defense of infinitesimals being absolute zeros. In 1755 Euler published his own investigation of the differentials in calculus by which he had attempted to show that the differentials or infinitesimals dx and dy are zeros and the differential coefficient $\frac{dy}{dx}$, a vanishing fraction, reduces to a finite variable: investigation which, at that time, seemed to favour the opinion of the opponents of the doctrine of zero divided by zero producing a finite quotient. His arguments, though very just, were, however, insufficient for proving that the differential coefficient of $y = x^n$, for any specified value n, is nx^{n-1}; for he wrote:

> Thus if it should be that $y = x^2$, in differential calculus it is shown to be $\frac{dy}{dx} = 2x$, and neither is this ratio of the increments true, unless the increment dx, from which dy arises, is placed equal to zero. Now nevertheless here in a true inquiry of differentials, the common talk about differentials as if they were absolute [quantities] can be tolerated, provided that always in the mind at least they can be referred to the truth. Hence we say correctly, if $y = x^2$, there could be $dy = 2xdx$, even if it is not false, if anyone says $dy = 3xdx$ or $dy = 4xdx$, as on account of $dx = 0$ and $dy = 0$ there equalities might stand also; but only the true ratio $\frac{dy}{dx} = 2x$ is to be agreed upon.

Many commentators do no appreciate Euler's explanation of the foundations of calculus. In fact, the mathematician J. Gray gave the following remark:

> At some point it should be admitted that Euler's attempts at explaining the foundations of calculus in terms of differentials, which are and are not zero, are dreadfully weak.

Between 1930s and 1940s, the English Bishop Berkeley fought the use of ratios of absolute nothings. In Query 31 of his *The Analyst*, he asks rhetorically:

> Where there are no increments, whether there can be ratio of increments? Whether nothings can be considered as proportional to real quantities? Or whether to talk of their proportions be not to talk nonsense?

Then in Query 40 he disapproved the idea of treating absolute nothing like a quantity:

> Whether it be not a general case or rule that one and the same coefficient dividing each product gives equal quotients? And yet whether such coefficient can be interpreted by 0 or nothing? Or whether anyone will say that if the equation $2 \times 0 = 5 \times 0$ be divided by 0 the quotients on both sides are equal? Whether therefore a case may not be general with respect to all quantities and yet not extend to nothings or include the case of nothing? And whether the bringing nothing under the notion of quantity may not have betrayed men into false reasoning?

Martin Ohm in 1828 argued that division by zero is unacceptable. He stated that if a is not zero, but b is zero, then the quotient a/b has no meaning, for the quotient multiplied by zero gives only zero and not a. From thence the doctrine of zero divided by zero was banished.

3 On the Division by Zero $1/0$

We fully appreciate the fact that this subject of division by zero is a most difficult part of analysis [16] and, accordingly, it will be dealt with at greater length in the pages that follows. We are not unmindful of the fact that the notion of division by zero has been the subject of much controversy. Nor do we forget that anything which has to do with division by zero requires to be handled with the utmost care and that mathematical arguments ought to be like diamonds very clear and perfect. Our appeal shall be to mathematical inquiry.

When first studying the subject of division by zero, one has the impression that everything in this topic is paradoxical and inconsistent. One will hardly suspect the existence of an exact quotient of division by zero. It is our intention here to institute the exact quotient of a special case of division by zero, namely the division of unity by zero $1/0$.

To clear the way for a logical and rational consideration of the subject, we must commence our inquiry with the actual birth place of the quotient of $1/0$– the factorial of integers [17]. We must, for the purposes of this work, extend the factorial function

$$n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1 \tag{3.1}$$

to include the factorials of zero and negative integers.

One of the most overwhelming aspects of mathematics is its seemingly endless diversity of patterns. Observe the following pattern of numbers [18]:

$$\frac{4!}{4} = 3!, \qquad \frac{3!}{3} = 2!, \qquad \frac{2!}{2} = 1!.$$

By a continued repetition of this pattern the values of $0!$ and $(-1)!$ may be found without recourse to any recurrence formula. This is the way mathematical inductionists predict the behavior of numbers.

In each case, the left side of the equation is the indicated ratio of the factorial of an integer to the integer itself, and the right side is the factorial of the integer less one. Inductive reasoning would suggest that the next two equations in this pattern are

$$\frac{1!}{1} = 0! \qquad \text{and} \qquad \frac{0!}{0} = (-1)!.$$

Equating these equations gives

$$(-1)! = \frac{0!}{0} = \frac{1}{0}.$$

Thus we conclude that the exact quotient of the fraction $1/0$ is $(-1)!$. If we had used the familiar recurrence relation $y! = y(y-1)!$ [17] and set $y = 0$, we would have arrived at the same result.

Numbers were originally devised to record ideas of real quantities. So when a symbol such as $(-1)!$ arises to which there is no corresponding real quantity we must ask ourselves why? Why does the symbol appear at all if there exists no real quantity associated with it? Often, the only way to answer a question of this sort is to accept the symbol as we did the imaginary number $\sqrt{-1}$ and carry on manipulating with it to see if any new ideas are forthcoming.

The equality $(-1)! = 1/0$ opens a wide field before us. Two points will claim our attention. First, $(-1)!$ is the multiplicative inverse of zero, and this suggests that $(-1)!$ is the Alter Ego of $1/0$. Second, $(-1)!$ is an infinite quantity as it is the infinite product of all the negative integers,

$$(-1)! = (-1) \cdot (-2) \cdot (-3) \cdots .$$

When we affirm that $1/0$ equals infinity and not undefined, we are not merely striving about words but insisting on a distinction that is of vital importance. We can reach the notion of infinity this way. Consider the relation

$$y = \frac{1}{x-1}.$$

If x is very nearly equal to 1 then y is very large and very nearly equal to infinity, and finally when x equals 1, y is infinite. Just here is the source of all confusions, errors and paradoxes. But the fact is that the quantity y is not infinite as long as $x - 1 \neq 0$, but when $x = 1$ or when $x - 1 = 0$, then

$$y = \frac{1}{0} = (-1)! = (-1) \cdot (-2) \cdot (-3) \cdots .$$

Observe that the symbol 0 is the absolute naught acting as a number.

Not only did Bhaskara himself speak on this subject of $a/0$ being infinite as we have mentioned in the previous sections, but other writers also weighed in heavily on this matter. The modern view of division by zero states that the operation of $a/0$ is meaningless or undefined. This doctrine cannot stand the test of certain physical phenomena.

The set of finite numbers is sufficient to represent very many practical situations, but it is unable to provide an answer for the following problem. If we want to determine the number $n(\theta)$ of images of an object placed between two plane mirrors inclined at an angle of θ, it is necessary to apply the experimental formula

$$n(\theta) = \frac{360}{\theta} - 1.$$

Using this formula, it is easily seen that $n(0.1) = 3599$, $n(0.01) = 35999$, $n(0.001) = 359999$, and so on. But if the two mirrors are set facing each other, i.e when $\theta = 0$, the number of images formed by these mirrors is

$$n(0) = \frac{360}{0} - 1.$$

What is $n(0)$? We cannot say that this is meaningless or undefined because of the division by zero $\frac{360}{0}$ for it is very clear that images are in reality formed in the two mirrors and we can locate and count them as far as we wish. This imports to us that division by zero is meaningful and defined. Since $1/0 = (-1)!$ we write

$$n(0) = 360 \times (-1)! - 1$$

which is an infinite quantity. Therefore we say that the number of images formed by the two parallel plane mirrors is infinitely great.

Let us here frame clear ideas of infinity. The term refers to that which is so large that it cannot be reached by counting or obtained by measurement. An infinite number of elements such as the images formed of an object placed between two plane mirrors facing each other is beyond counting because there is no last element. An infinite amount of something is above or without measure because it is unlimited in size. Thus infinite time or eternity is beyond spending and infinite length or distance is above coverage. We may say here that infinity, whenever it appears, is a sign of an unaccomplished task.

3.1 Factorials of negative integers expressed in terms of $(-1)!$

It is proper to show how the factorial of every other negative integer is connected to $(-1)!$. We begin with the recurrence relation

$$n! = n \cdot (n-1)!.$$

If we now substitute $-n$ for n in this relation, n being a positive integer, we have the result

$$(-n)! = (-n) \cdot (-n-1)!$$

which, when expressed as

$$(-n)! = (-n) \cdot (-n-1) \cdot (-n-2) \cdots (-n-m)!$$

is transformed into

$$(-n-m)! = \frac{(-n)!}{(-n) \cdot (-n-1) \cdot (-n-2) \cdots (-n-m+1)}$$

where $m = 1, 2, 3, \ldots$. Letting $n = 1$, we get

$$(-1-m)! = \frac{(-1)!}{(-1) \cdot (-2) \cdot (-3) \cdots (-m)}$$

which becomes

$$\frac{(-m)!}{(-m)} = \frac{(-1)!}{(-1) \cdot (-2) \cdot (-3) \cdots (-m)}$$

which in its own turn becomes

$$(-m)! = \frac{(-1)!}{(-1) \cdot (-2) \cdot (-3) \cdots (-(m-1))}.$$

This is then simplified to the interesting form

$$(-m)! = \frac{(-1)!}{(-1)^{m-1}(m-1)!}. \tag{3.2}$$

Thus we are here presented with a very interesting relation, the design of which appears to show the relation in which $(-1)!$ links with every other negative integer factorial, indicating that $(-1)!$ could be looked on as the head and forefront of all the negative integers factorials. Furthermore, this relation shows that the factorial of every negative integer is infinite, larger than any fixed assignable quantity. If we let $m = 2, 3, 4, \ldots$, we obtain the factorials of the negative integers in terms of the infinite number $(-1)!$ as follows:

$$(-2)! = \frac{(-1)!}{(-1)^{2-1}(2-1)!} = -(-1)! = -\frac{1}{0}$$
$$(-3)! = \frac{(-1)!}{(-1)^{3-1}(3-1)!} = \frac{1}{2!}(-1)! = \frac{1}{2!}\cdot\frac{1}{0}$$
$$(-4)! = \frac{(-1)!}{(-1)^{4-1}(4-1)!} = -\frac{1}{3!}(-1)! = -\frac{1}{3!}\cdot\frac{1}{0}$$
$$(-5)! = \frac{(-1)!}{(-1)^{5-1}(5-1)!} = \frac{1}{4!}(-1)! = \frac{1}{4!}\cdot\frac{1}{0}$$

and so forth. It is our intention now to draw conclusions from these. It can be easily seen that the factorial of a negative even integer equals the negative of minus one factorial divided by the factorial of the even integer less one. On the other hand, the factorial of a negative odd integer equals the positive of minus one factorial divided by the factorial of the odd integer less one.

4 Justifications

4.1 Series expansion of $\ln(1+x)$

We start with the expansion

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots .$$

Letting $\alpha = 0$ gives

$$(1+x)^0 = 1 + 0x + \frac{0(0-1)}{2!}x^2 + \frac{0(0-1)(0-2)}{3!}x^3 + \cdots \tag{4.1}$$

which actually becomes

$$(1+x)^0 = 1$$

since the rest terms are all absolute nothing. But because further operations impend, we must strictly uphold (4.1) and use it in these operations.

Now consider the series expansion

$$a^y = 1 + y\ln a + \frac{y^2\ln^2 a}{2!} + \cdots .$$

Setting $a = 1 + x$ gives

$$(1+x)^y = 1 + y\ln(1+x) + \frac{y^2\ln^2(1+x)}{2!} + \cdots$$

and letting $y = 0$ gives

$$(1+x)^0 = 1 + 0\ln(1+x) + \frac{0^2\ln^2(1+x)}{2!} + \cdots . \tag{4.2}$$

Equating (4.1) and (4.2) furnishes

$$1 + 0x + \frac{0(0-1)}{2!}x^2 + \frac{0(0-1)(0-2)}{3!}x^3 + \cdots = 1 + 0\ln(1+x) + \frac{0^2\ln^2(1+x)}{2!} + \cdots$$

which becomes

$$0x + \frac{0(0-1)}{2!}x^2 + \frac{0(0-1)(0-2)}{3!}x^3 + \cdots = 1 - 1 + 0\ln(1+x) + \frac{0^2\ln^2(1+x)}{2!} + \cdots .$$

The expression $1-1$ is not worthy of a notational symbol as it is the elimination of 1 [See the paper [19] for details]. Thus we write

$$0x + \frac{0(0-1)}{2!}x^2 + \frac{0(0-1)(0-2)}{3!}x^3 + \cdots = 0\ln(1+x) + \frac{0^2\ln^2(1+x)}{2!} + \cdots .$$

Let us now divide both sides of the above equations by 0. Accomplishing this, we have

$$x + \frac{(0-1)}{2!}x^2 + \frac{(0-1)(0-2)}{3!}x^3 + \cdots = \ln(1+x) + \frac{0\ln^2(1+x)}{2!} + \cdots .$$

Since this is the terminus of our calculation, we make 0 display its unique feature of being void. Hence, the equation above becomes

$$x + \frac{(-1)}{2!}x^2 + \frac{(-1)(-2)}{3!}x^3 + \cdots = \ln(1+x)$$

which, being simplified and rearranged, becomes

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots .$$

4.2 Series Expansion of $(1+z)^{-1}$

A well known identity in combinatorics is as follows:

$$(1+z)^n = \sum_{k=0}^{\infty} {}_nC_k z^k$$

We use the symbol 0 for only evaluation processes e.g when we say let $x = 0$ in any expression of x or when we set $x = c$ in the expression $x - c$, *viz* $(c) - c = 0$ or when we put $x = -c$ in the expression $x + c$ *viz* $(-c) + c = 0$. It follows that setting $x = c$ in the expression $c - x$, which is also $-(x-c)$, is -0 and putting $x = -c$ in the expression $-x - c$, which is also $-(x + c)$, is -0. The zeros 0 and -0 coincide at the origin of the number line and hence signify the same physical thing, absolute nothing. Still in impending operations one cannot be substituted for the other. This last statement is true also of other zeros such as $2 \cdot 0$, 0^4, etc. Elimination expressions such as $1-1$, $2-2$, $x - x$, $3x^2 - 3x^2$, etc which arises only during simplification of algebraic or arithmetic expressions or equations are considered blank.

The reader should note that

$$\frac{a \cdot 0^m}{0^n} = a \cdot 0^{m-n}.$$

See the paper [19] for details.

where ${}_nC_k$ is a combination function [20]. If we substitute -1 for n and evaluate, we get the following workings:

$$\begin{aligned}
(1+z)^{-1} &= \sum_{k=0}^{\infty} {}_{-1}C_k z^k \\
&= {}_{-1}C_0 z^0 + {}_{-1}C_1 z^1 + {}_{-1}C_2 z^2 + {}_{-1}C_3 z^3 + \dots \\
&= \frac{(-1)!}{0!(-1)!} z^0 + \frac{(-1)!}{1!(-2)!} z^1 + \frac{(-1)!}{2!(-3)!} z^2 + \frac{(-1)!}{3!(-4)!} z^3 + \dots \\
&= \frac{1}{0!} z^0 + \frac{1/0}{1!(-(1/0))} z^1 + \frac{1/0}{2!\left(\frac{1}{2!}(1/0)\right)} z^2 + \frac{1/0}{3!\left(\frac{-1}{3!}(1/0)\right)} z^3 + \dots \\
&= \frac{1}{0!} z^0 + \frac{0}{1!(-0)} z^1 + \frac{0}{2!\left(\frac{1}{2!}\cdot 0\right)} z^2 + \frac{0}{3!\left(\frac{-1}{3!}\cdot 0\right)} z^3 + \dots \\
&= z^0 - z^1 + z^2 - z^3 + \dots
\end{aligned}$$

4.3 Series expansion of $\ln(1+y/x)$

This requires the use of the result of $\left(b^0 - a^0\right)/0$ and we therefore inquire into this. We begin with the identity

$$a^x = 1 + x\ln a + \frac{(x\ln a)^2}{2!} + \cdots .$$

Therefore the difference $b^x - a^x$ is expressed as the following:

$$b^x - a^x = 1 - 1 + (\ln b - \ln a)\,x + \frac{\left(\ln^2 b - \ln^2 a\right)x^2}{2!} + \cdots$$

which, considering $1-1$ as blank since it is the elimination of 1, becomes

$$b^x - a^x = (\ln b - \ln a)\,x + \frac{\left(\ln^2 b - \ln^2 a\right)x^2}{2!} + \cdots .$$

Letting $x = 0$ gives

$$b^0 - a^0 = (\ln b - \ln a)\,0 + \frac{\left(\ln^2 b - \ln^2 a\right)0^2}{2!} + \cdots .$$

Dividing both sides by 0 furnishes

$$\frac{b^0 - a^0}{0} = (\ln b - \ln a) + \frac{\left(\ln^2 b - \ln^2 a\right)0}{2!} + \cdots$$

which becomes the final result

$$\frac{b^0 - a^0}{0} = \ln b - \ln a \tag{4.3}$$

since

$$\frac{\left(\ln^2 b - \ln^2 a\right)0}{2!} + \cdots$$

is equal to naught.

We now consider the series expansion of $\ln(1+y/x)$. We start with the binomial theorem

$$\begin{aligned}
(x+y)^n = x^n &+ \frac{n!}{1!(n-1)!}x^{n-1}y + \frac{n!}{2!(n-2)!}x^{n-2}y^2 \\
&+ \frac{n!}{3!(n-3)!}x^{n-3}y^3 + \dots
\end{aligned}$$

which can be rewritten as

$$(x+y)^n - x^n = \frac{n!}{1!(n-1)!}x^{n-1}y + \frac{n!}{2!(n-2)!}x^{n-2}y^2 + \frac{n!}{3!(n-3)!}x^{n-3}y^3 + \dots$$

Letting $n = 0$ gives

$$(x+y)^0 - x^0 = \frac{0!}{1!(0-1)!}x^{0-1}y + \frac{0!}{2!(0-2)!}x^{0-2}y^2 + \frac{0!}{3!(0-3)!}x^{0-3}y^3 + \dots$$

which becomes

$$(x+y)^0 - x^0 = \frac{1}{1!(-1)!}x^{-1}y + \frac{1}{2!(-2)!}x^{-2}y^2 + \frac{1}{3!(-3)!}x^{-3}y^3 + \dots.$$

It was demonstrated that

$$(-1)! = \frac{1}{0}$$
$$(-2)! = -\frac{1}{1!}\cdot\frac{1}{0}$$
$$(-3)! = \frac{1}{2!}\cdot\frac{1}{0}$$
$$(-4)! = -\frac{1}{3!}\cdot\frac{1}{0}$$
$$(-5)! = \frac{1}{4!}\cdot\frac{1}{0}$$

and so on. Applying these we get

$$(x+y)^0 - x^0 = \frac{0}{1!}x^{-1}y + \frac{-1!\cdot 0}{2!}x^{-2}y^2 + \frac{2!\cdot 0}{3!}x^{-3}y^3 + \dots$$

which becomes

$$(x+y)^0 - x^0 = \frac{0}{1}x^{-1}y - \frac{0}{2}x^{-2}y^2 + \frac{0}{3}x^{-3}y^3 - \dots$$

which in turn, dividing by 0, becomes

$$\frac{(x+y)^0 - x^0}{0} = \frac{y}{x} - \frac{1}{2}\left(\frac{y}{x}\right)^2 + \frac{1}{3}\left(\frac{y}{x}\right)^3 - \dots.$$

Thus, applying (4.3), we obtain

$$\ln(x+y) - \ln x = \frac{y}{x} - \frac{1}{2}\left(\frac{y}{x}\right)^2 + \frac{1}{3}\left(\frac{y}{x}\right)^3 - \dots$$

which becomes

$$\ln\left(\frac{x+y}{x}\right) = \frac{y}{x} - \frac{1}{2}\left(\frac{y}{x}\right)^2 + \frac{1}{3}\left(\frac{y}{x}\right)^3 - \dots$$

which finally becomes

$$\ln\left(1+\frac{y}{x}\right) = \frac{y}{x} - \frac{1}{2}\left(\frac{y}{x}\right)^2 + \frac{1}{3}\left(\frac{y}{x}\right)^3 - \dots.$$

4.4 Sum of alternating harmonic series

In one of his works, Euler showed that

$$1^n - 2^n + 3^n - 4^n + \cdots = (-1)^s \, \frac{2^{n+1}-1}{n+1} B_{n+1}$$

where $s = [(n+1)/2]$ is the integer part of $(n+1)/2$ and B_n is the nth Bernoulli number [21]. Setting $n = -1$ gives

$$1^{-1} - 2^{-1} + 3^{-1} - 4^{-1} + \cdots = (-1)^{[(-1+1)/2]} \, \frac{2^{-1+1}-1}{-1+1} B_{-1+1}$$

which simplifies to

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = (-1)^{[(0)/2]} \, \frac{2^0 - 1^0}{0} B_0.$$

With the understanding that $B_0 = 1$ and considering that

$$\frac{2^0 - 1^0}{0} = \ln 2 - \ln 1 = \ln 2,$$

we get

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2.$$

4.5 Integration of x^{-1}

The ordinary rule for the integration of x^n is known to be

$$\int_a^b x^n dx = \left[\frac{x^{n+1}}{n+1}\right]_a^b.$$

One instance stands out and sometimes gives trouble though being wrongly regarded as an exceptional case - a sort of black sheep - of the x^n family. This instance is x^{-1} which on application of the ordinary rule for the integration of x^n gives

$$\int_a^b x^{-1} dx = \left[\frac{x^0}{0}\right]_a^b$$

To resolve this difficulty, we work as follows. We evaluate the right-hand expression of the above identity and get

$$\int_a^b x^{-1} dx = \frac{b^0 - a^0}{0}$$

which, considering (4.3), becomes

$$\int_a^b x^{-1} dx = \ln b - \ln a = [\ln x]_a^b.$$

4.6 Euler's constant

Here we use the way of zero and infinity to derive the famous identity

$$\gamma = H_\Omega - \ln \Omega$$

where Ω is the infinite integer for which $H_\Omega = 1 + 1/2 + 1/3 + \cdots + 1/\Omega$ is the harmonic series and $\gamma = 0.57721\ldots$ is the Euler's constant. We start with the sums of powers formula

$$\sum_{k=1}^{n} k^m = \frac{(B+n+1)^{m+1} - B^{m+1}}{m+1}$$

where B^m equals the mth Bernoulli number B_m. If we set $m = -1$, we get

$$\sum_{k=1}^{n} k^{-1} = \frac{(B+n+1)^{-1+1} - B^{-1+1}}{-1+1}$$

which becomes

$$\sum_{k=1}^{n} k^{-1} = \frac{(B+n+1)^{0} - B^{0}}{0}.$$

If we consider (4.3), the above result becomes

$$\sum_{k=1}^{n} \frac{1}{k} = \ln(B+n+1) - \ln B = \ln\left(\frac{B+n+1}{B}\right) = \ln\left(1 + \frac{n+1}{B}\right).$$

This, setting $\sum_{k=1}^{n} \frac{1}{k} = H_n$ where H_n is the nth harmonic number, becomes

$$H_n = \ln\left(\frac{B+n+1}{B}\right) = \ln\left(1 + \frac{n+1}{B}\right). \tag{4.4}$$

The question now is, What is B? To answer this question we need to express B in terms of n and set $n = 1, 2, 3, \ldots$ to see what would happen to B. Now B expressed as the subject is

$$B = \frac{n+1}{e^{H_n} - 1}$$

where e is Euler's number. Computing B for the first few values of n, we observe that B varies with n. We conclude that B is a variable depending on n. Let now B be rewritten as the function $B(n)$. The above equation becomes

$$B(n) = \frac{n+1}{e^{H_n} - 1}.$$

It remains to compute the functional value of $B(n)$ when H_n becomes the harmonic series $H_\Omega = 1 + 1/2 + 1/3 + \cdots + 1/\Omega$. To perform this we construct a table of values of $B(n)$ as n becomes larger and larger.

Table 1. Values of $B(n)$

n	H_n	$B(n)$
10	2.92896 . . .	0.62117 . . .
100	5.18737 . . .	0.56742 . . .
1000	7.48547 . . .	0.56205 . . .
10000	9.78760 . . .	0.56151 . . .
100000	12.0901 . . .	0.56146 . . .
1000000	14.3927 . . .	0.56146 . . .
⋮	⋮	⋮

From this table, we see that as n becomes larger and larger, $B(n)$ becomes closer and closer to $0.561459\ldots = e^{-\gamma}$ where $e = 2.71828\ldots$. Let Ω be the value of n for which H_Ω is the harmonic series. It follows that

$$B(\Omega) = 0.561459\ldots = e^{-\gamma}.$$

We see immediately that, setting $n = \Omega$ in (4.4), the harmonic number becomes

$$H_\Omega = \ln\left(1 + \frac{\Omega+1}{e^{-\gamma}}\right)$$

which becomes

$$H_\Omega = \ln\left(1 + (\Omega+1)e^\gamma\right).$$

Since Ω is an infinite quantity, it is immutable in the presence of finite addends or minuends. Thus the above equation becomes

$$H_\Omega = \ln\left(\Omega e^\gamma\right)$$

which becomes the required identity

$$\gamma = H_\Omega - \ln\Omega.$$

4.7 Generalizing a family of figurate numbers

A family of figurate numbers defined as

$$P_r(n) = \binom{n+r-1}{r}$$

is well worthy of our attention. One enchanting feature of these figurate numbers is that if the nth term of a sequence of any given r- figurate numbers be added to the $(n+1)$th term of the sequence of the preceding r- figurate numbers, the sum will be equal to the $(n+1)$th term of the sequence of the given r- figurate numbers. As an instance, let us take two sequences of the triangular numbers and the tetrahedral numbers:

$$\begin{array}{cccccc} 1, & 3, & 6, & 10, & 15, & \dots \\ 1, & 4, & 10, & 20, & 35, & \dots. \end{array}$$

Here, if we add to any term in the upper sequence that term in the lower which stands one place to the left, the sum is the next term in the lower sequence. Starting with 6 sequences of 1's, all of the sequences of figurate numbers may be deduced in succession by the aid of this principle:

$$\begin{array}{ccccccccccc} r=0 & : & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ r=1 & : & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ r=2 & : & 1 & 3 & 6 & 10 & 15 & 21 & 28 & 36 \\ r=3 & : & 1 & 4 & 10 & 20 & 35 & 56 & 84 & 120 \\ r=4 & : & 1 & 5 & 15 & 35 & 70 & 126 & 210 & 330 \\ r=5 & : & 1 & 6 & 21 & 56 & 126 & 252 & 462 & 792. \end{array}$$

By the just principle laid down by Bhaskara that $a \cdot 0/0 = a$ we shall generalize these figurate numbers. Let us now consider the cases where r is a negative integer. We begin by setting $r = -m$, that is

$$P_{-m}(n) = \binom{n{-}m-1}{-m}$$

which becomes

$$P_{-m}(n) = \binom{n-m-1}{-m}.$$

This binomial coefficient is expressible as

$$\begin{aligned} P_{-m}(n) &= \frac{(n-m-1)!}{(-m)!(n-1)!} \\ &= \frac{(n-m-1)!}{\left(\frac{1}{(-1)^{m-1}(m-1)!}\cdot\frac{1}{0}\right)\cdot(n-1)!} \\ &= \frac{(-1)^{m-1}(m-1)!(n-m-1)!\cdot 0}{(n-1)!}. \end{aligned}$$

Letting $m = 1$, we get

$$P_{-1}(n) = \frac{(-1)^{1-1}(1-1)!(n-1-1)! \cdot 0}{(n-1)!}$$

which becomes

$$\begin{aligned} P_{-1}(n) &= \frac{0!(n-2)! \cdot 0}{(n-1)!} \\ &= \frac{0}{n-1}. \end{aligned}$$

Setting $n = 1, 2, 3, \ldots$ gives

$$\begin{aligned} P_{-1}(1) &= \frac{0}{1-1} = \frac{0}{0} = 1 \\ P_{-1}(2) &= \frac{0}{2-1} = \frac{0}{1} \\ P_{-1}(3) &= \frac{0}{3-1} = \frac{0}{2} \\ P_{-1}(4) &= \frac{0}{4-1} = \frac{0}{3} \\ P_{-1}(5) &= \frac{0}{5-1} = \frac{0}{4} \end{aligned}$$

and so on. The case where $m = 1$ consists of only one finite figurate number, namely, 1. The rest numbers are all absolute nothing.

Letting $m = 2$, we get

$$P_{-2}(n) = \frac{(-1)^{2-1}(2-1)!(n-2-1)! \cdot 0}{(n-1)!}$$

which becomes

$$\begin{aligned} P_{-2}(n) &= \frac{-1!(n-3)! \cdot 0}{(n-1)!} \\ &= \frac{-0}{(n-1)(n-2)}. \end{aligned}$$

Setting $n = 1, 2, 3, \ldots$ gives

$$\begin{aligned} P_{-2}(1) &= \frac{-0}{(1-1)(1-2)} = \frac{-0}{(0)(-1)} = 1 \\ P_{-2}(2) &= \frac{-0}{(2-1)(2-2)} = \frac{-0}{(1)(0)} = -1 \\ P_{-2}(3) &= \frac{-0}{(3-1)(3-2)} = \frac{-0}{(2)(1)} = \frac{-0}{2} \\ P_{-2}(4) &= \frac{-0}{(4-1)(4-2)} = \frac{-0}{(3)(2)} = \frac{-0}{6} \\ P_{-2}(5) &= \frac{-0}{(5-1)(5-2)} = \frac{-0}{(4)(3)} = \frac{-0}{12} \end{aligned}$$

and so on. For the case where $m = 2$ there exists only two finite figurate numbers, namely, 1 and -1. The other numbers are all absolute nothing.

Letting $m = 3$, we get

$$P_{-3}(n) = \frac{(-1)^{3-1}(3-1)!(n-3-1)! \cdot 0}{(n-1)!}$$

which becomes

$$P_{-3}(n) = \frac{2!(n-4)! \cdot 0}{(n-1)!} = \frac{2 \cdot 0}{(n-1)(n-2)(n-3)}.$$

Setting $n = 1, 2, 3, \ldots$ gives

$$P_{-3}(1) = \frac{2 \cdot 0}{(1-1)(1-2)(1-3)} = \frac{2 \cdot 0}{(0)(-1)(-2)} = 1$$
$$P_{-3}(2) = \frac{2 \cdot 0}{(2-1)(2-2)(2-3)} = \frac{2 \cdot 0}{(1)(0)(-1)} = -2$$
$$P_{-3}(3) = \frac{2 \cdot 0}{(3-1)(3-2)(3-3)} = \frac{2 \cdot 0}{(2)(1)(0)} = 1$$
$$P_{-3}(4) = \frac{2 \cdot 0}{(4-1)(4-2)(4-3)} = \frac{2 \cdot 0}{(3)(2)(1)} = \frac{0}{3}$$
$$P_{-3}(5) = \frac{2 \cdot 0}{(5-1)(5-2)(5-3)} = \frac{2 \cdot 0}{(4)(3)(2)} = \frac{0}{12}$$

and so on. The case where $m = 3$ comprises three finite figurate numbers which are 1, -2 and 1. The others are all naught.

The reader can apply the same approach for cases where $m = 4, 5, 6, \ldots$. If he gathers together all the finite figurate numbers for all the cases, he will discover that they form signed Pascal Triangle numbers. We thus generalize the figurate numbers by combining the old and new sequences of figurate numbers. The table below shows these sequences.

$r=-6$	:	1	-5	10	-10	5	-1		
$r=-5$	:	1	-4	6	-4	1			
$r=-4$	:	1	-3	3	-1				
$r=-3$	:	1	-2	1					
$r=-2$	:	1	-1						
$r=-1$	:	1							
$r=0$	:	1	1	1	1	1	1	1	1
$r=1$	:	1	2	3	4	5	6	7	8
$r=2$	:	1	3	6	10	15	21	28	36
$r=3$	:	1	4	10	20	35	56	84	120
$r=4$	:	1	5	15	35	70	126	210	330
$r=5$	:	1	6	21	56	126	252	462	792.

5 Conclusion

We discussed the operation of division by zero in Bhaskara's framework. We showed that his principle of impending operation on zero is infallible by giving a good number of examples to illustrate it. We also demonstrated that the simplest case of division by zero $1/0$ equals minus one factorial.

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