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Hyers-Ulam Stability of a Quadratic Functional Equation in Banach Algebras: A Fixed Point Approach

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Authors' contributions

All the authors contributed equally and significantly in writing this paper, participated in its design and coordination, participated in the sequence alignment. All authors read and approved the final manuscript.

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Abstract

In this paper, we obtain the general solution of a quadratic functional equation and study the Hyers-Ulam stability of generalized derivations in Banach algebras using a fixed point approach.

Keywords: Quadratic functional equation; generalized Hyers-Ulam stability; Banach algebra; fixed point.

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1 Introduction

The stability problem of functional equation originated from a question of S.M. Ulam [1] . In 1940, S. M. Ulam gave the following question concerning the stability of homomorphisms:

Let $(G, \cdot)$ be a group and let H be a metric group with metric $d(.,.)$. Given $\epsilon > 0$ does there exist a $\delta > 0$ such that if a function $f : G \to H$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$ then a homomorphism $a : G \to H$ exists with $d(f(x), a(x)) < \epsilon$ for all $x \in G$?

In other words, under what condition does there exist a homomorphism near an approximate homomorphism?. D.H. Hyers [2] gave an affirmative answer to the problem of Ulam under the assumption that the groups are Banach spaces. The theorem of Hyers was extended by T. Aoki [3] for approximately additive mappings and by Th.M.Rassias [4] for approximately linear mappings. Four years later, J.M. Rassias [5] by applying Th. M. Rassias's approach [4] for the stability of mappings when the norm of the Cauchy difference is bounded by the sum of powers of norms, obtained a similar theorem in which the norm of the Cauchy difference is bounded by the product of powers of norms.

In 2009, Jung Rye Lee and Choonkil Park [6] proved the generalized Hyers-Ulam Stability of homomorphism and of derivations on Banach algebras using fixed point method for the 3-variable Cauchy functional equation

$$f(x + y + z) = f(x) + f(y) + f(z)$$

In 2010, C.Park and A.Najati [7] investigated Hyers-Ulam-Rassias-Stability of homomorphism and of derivations in Banach algebras associated with generalized additive functional inequality

$$\|af(x) + bf(y) + cf(z)\| \leq \|f(\alpha x + \beta y + \gamma z)\|$$

In 2012, Yeol Je Cho, Jung IM Kang and Reza Saadati [8] investigated generalized Hyers-Ulam stability of homomorphisms and of derivations on Banach algebras using fixed point method for the following additive functional equation

$$\sum_{i=1}^{m} f\left(mx_i + \sum_{j=1, j\neq i}^{m} x_j\right) + f\left(\sum_{i=1}^{m} x_i\right) = 2f\left(\sum_{i=1}^{m} mx_i\right)$$

for all $m \in \mathbb{N}$ with $m \geq 2$.

The stability problems of various functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see[9]-[15],[16],[17], [18],[19]-[34]).

Definition 1.1. Let X be a set. A function $d : X \times X \to [0, \infty)$ is called generalized metric on X if d satisfies

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

In 1996, Isac and Rassias [35] were the first to provide applications of stability theory of functional equations for the new fixed point theorems. By using fixed point methods, several stability problems have been extensively investigated by number of authors.

Now we shall recall a fundamental result in fixed point theory.

Theorem 1.1. *Let (X,d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either $d(J^n x, J^{n+1}x) = \infty$ for all nonnegative integers n or there exists a positive integer n_0 such that*

1. $d(J^n x, J^{n+1}x) < \infty, \forall n \geq n_0$;.
2. *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of J;*
3. y^* *is the unique fixed point of J in the set* $Y = \{y \in X | d(J^{n_0}x, y) < \infty\}$;
4. $d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ *for all* $y \in Y$.

In this paper, we obtain the general solution of a quadratic functional equation

$$\begin{aligned} &f(2x-y)+f(2y-z)+f(2z-x)+f(x+y+z)-f(x-y+z)\\ &-f(x+y-z)-f(x-y-z)=3f(x)+3f(y)+3f(z) \end{aligned} \tag{1.1}$$

and investigate its Hyers-Ulam stability in Banach algebras using fixed point approach. This paper is organized as follows.

In Section: 2, we provide the general solution of a quadratic functional equation (1.1). In Section:3, we prove the Hyers-Ulam-Rassias stability of homomorphisms for the quadratic functional equation (1.1) in real Banach algebras and in Section: 4, we investigate the Hyers-Ulam stability of the functional equation (1.1). In Section:5, we prove the Hyers-Ulam-Rassias stability of generalized derivations on real Banach algebras for the quadratic functional equation (1.1) by using fixed point method.

2 The General Solution

The following theorem provide the general solution of the functional equation (1.1) by establishing a connection with the classical quadratic functional equation.

Theorem 2.1. *Let X and Y be real vector spaces. A function $f : X \to Y$ satisfies the functional equation*

$$\begin{aligned} &f(2x-y)+f(2y-z)+f(2z-x)+f(x+y+z)-f(x-y+z)\\ &-f(x+y-z)-f(x-y-z)=3f(x)+3f(y)+3f(z) \end{aligned} \tag{2.1}$$

for all $x, y, z \in X$ if and only if it satisfies the quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{2.2}$$

for all $x \in X$.

Proof. Suppose a function $f : X \to Y$ satisfies (2.1). Setting $(x, y, z) = (x, x, x)$ in (2.1), we obtain

$$f(x)+f(x)+f(x)+f(3x)-f(x)-f(x)-f(-x)=3f(x)+3f(x)+3f(x)$$

which gives

$$f(3x)=8f(x)+f(-x). \tag{2.3}$$

Again, setting $(x, y, z) = (-x, x, x)$ in (2.1), we obtain

$$\begin{aligned} &f(-3x)+f(x)+f(3x)+f(x)-f(-x)-f(-x)-f(-3x)\\ &=3f(-x)+3f(x)+3f(x) \end{aligned}$$

which gives

$$f(3x)=5f(-x)+4f(x). \tag{2.4}$$

Equating (2.3) and (2.4) we obtain $f(x) = f(-x)$. Thus, f is an even function. Putting $x = 0$ in equation (2.1), we obtain $f(0) = 0$. Setting $(x, y, z) = (x, 0, 0)$ in (2.1) and using evenness we arrive,

$$f(2x) - f(x) = 3f(x),$$

which gives

$$f(2x) = 4f(x). \tag{2.5}$$

In equation (2.3), using the evenness, we arrive

$$f(3x) = 9f(x) \quad forall \quad x \in X. \tag{2.6}$$

Extending this ideas, in general we obtain $f(nx) = n^2 f(x)$. Setting $x = 0$ in equation (2.1), using (2.5) and evenness of f, we obtain

$$f(2y - z) - f(-y + z) - f(y - z) = 2f(y) - f(z). \tag{2.7}$$

Again, setting $(y, z) = (y, y - z)$ in equation (2.7) and using evenness of f, we obtain

$$f(y + z) + f(y - z) = 2f(y) + 2f(z).$$

Setting $(y, z) = (x, y)$, we obtain

$$f(x + y) + f(x - y) = 2f(x) + 2f(y).$$

Suppose that a function $f : X \to Y$ satisfies (2.2). Setting $(x, y) = (x + y, z), (y + z, x),$ $(z + x, y)$ respectively in equation (2.2), we obtain a set of equations:

$$\begin{aligned} f(x + y + z) + f(x + y - z) &= 2f(x + y) + 2f(z) \\ f(y + z + x) + f(y + z - x) &= 2f(y + z) + 2f(x) \\ f(z + x + y) + f(z + x - y) &= 2f(z + x) + 2f(y). \end{aligned} \tag{2.8}$$

Again setting $(x, y) = (x, y - z), (y, z - x), (z, x - y)$ respectively in equation (2.2), we have another set of equations:

$$\begin{aligned} f(x + y - z) + f(x - y + z) &= 2f(x) + 2f(y - z) \\ f(y + z - x) + f(y - z + x) &= 2f(y) + 2f(z - x) \\ f(z + x - y) + f(z - x + y) &= 2f(z) + 2f(x - y). \end{aligned} \tag{2.9}$$

Subtracting half the sum of all equations in (2.9) from the sum of all equations in (2.8), we obtain

$$\begin{aligned} 3f(x + y + z) = {} & 2[f(x + y) + f(y + z) + f(z + x)] - [f(x - y) + f(y - z) + f(z - x)] \\ & + [f(x) + f(y) + f(z)]. \end{aligned} \tag{2.10}$$

If we rewrite (2.2) as $f(x + y) = 2f(x) + 2f(y) - f(x - y)$ and perform cyclic permutation of all variables, then (2.10) simplifies to

$$3f(x + y + z) = 9[f(x) + f(y) + f(z)] - 3[f(x - y) + f(y - z) + f(z - x)]. \tag{2.11}$$

Setting $(x, y) = (x, x - y)$ and all cyclic permutations of variables in (2.2), we have

$$\begin{aligned} f(2x - y) + f(y) &= 2f(x) + 2f(x - y), \\ f(2y - z) + f(z) &= 2f(y) + 2f(y - z), \\ f(2z - x) + f(x) &= 2f(z) + 2f(z - x). \end{aligned} \tag{2.12}$$

from equation (2.12), we get

$$\begin{aligned}&2\left[f\left(x-y\right)+f\left(y-z\right)+f\left(z-x\right)\right]\\&=f\left(2x-y\right)+f\left(2y-z\right)+f\left(2z-x\right)-\left[f\left(x\right)+f\left(y\right)+f\left(z\right)\right].\end{aligned} \tag{2.13}$$

Using (2.13) in equation (2.11), we get

$$f\left(2x-y\right)+f\left(2y-z\right)+f\left(2z-x\right)+2f\left(x+y+z\right)=7f\left(x\right)+7f\left(y\right)+7f\left(z\right). \tag{2.14}$$

Setting y by $y+z$ and $y-z$ in equation (2.2), we get

$$f\left(x+y+z\right)+f\left(x-y-z\right)=2f\left(x\right)+2f\left(y+z\right) \tag{2.15}$$

$$f\left(x+y-z\right)+f\left(x-y+z\right)=2f\left(x\right)+2f\left(y-z\right) \tag{2.16}$$

Adding equation (2.15) and (2.16), using (2.2) we get,

$$f\left(x+y+z\right)+f\left(x-y-z\right)+f\left(x+y-z\right)+f\left(x-y+z\right)=4f\left(x\right)+4f\left(y\right)+4f\left(z\right). \tag{2.17}$$

Subtracting equation (2.17) from (2.14), we obtain

$$\begin{aligned}&f(2x-y)+f(2y-z)+f(2z-x)+f(x+y+z)-f(x-y+z)\\&-f(x+y-z)-f(x-y-z)=3f(x)+3f(y)+3f(z)\end{aligned}$$

□

3 Stability of Homomorphisms in Real Banach Algebras

Throughout this section, assume that A is a real Banach algebra with norm $\|\cdot\|_A$ and that B is a real Banach algebra with norm $\|\cdot\|_B$. For a given mapping $f : A \to B$, we define

$$\begin{aligned}Cf(x,y,z):=\ &f(2x-y)+f(2y-z)+f(2z-x)+f(x+y+z)-f(x-y+z)\\&-f(x+y-z)-f(x-y-z)-3f(x)-3f(y)-3f(z)\end{aligned} \tag{3.1}$$

for all $x, y, z \in A$.

Note that a $\mathbb{C}$-linear mapping $H : A \to B$ is called a *algebra homomorphism* in Banach algebras if H satisfies $H(xy) = H(x)H(y)$ for all $x, y \in A$

We prove the Hyers-Ulam-Rassias stability of homomorphisms in real Banach algebras for the functional equation $Cf(x, y, z) = 0$.

Theorem 3.1. *Let $f : A \to B$ be a mapping for which there exists a function $\phi : A^3 \to [0,\infty)$ such that*

$$\sum_{j=0}^{\infty}\frac{1}{4^j}\phi(2^jx,2^jy,2^jz)<\infty \tag{3.2}$$

$$\|Cf(x,y,z)\|_B\le\phi(x,y,z) \tag{3.3}$$

$$\|f(xy)-f(x)f(y)\|_B\le\phi(x,y,0) \tag{3.4}$$

for all $x, y, z \in A$. If there exists an $L < 1$ such that $\phi(x, 0, 0) \le 4L\phi\left(\frac{x}{2}, 0, 0\right)$ for all $x \in A$ and if $f(tx)$ is continuous in $t \in \Re$ for each fixed $x \in A$, then there exists a unique homomorphism $H : A \to B$ such that

$$\|f(x)-H(x)\|_B\le\frac{1}{4-4L}\phi(x,0,0) \tag{3.5}$$

for all $x \in A$.

Proof. Consider the set

$$X := \{g : A \to B\} \tag{3.6}$$

and introduce the generalized metric on X:

$$d(g,h) = inf\left\{C \in \Re_+ : \|g(x) - h(x)\|_B \le C\phi(x,0,0), \forall x \in A\right\}. \tag{3.7}$$

It is easy to show that (X, d) is complete. Now, we consider the linear mapping $J : X \to X$ such that

$$Jg(x) := \frac{1}{4}g(2x) \tag{3.8}$$

for all $x \in A$. By [[12], Theorem 3.1]

$$d(Jg, Jh) \le Ld(g,h) \tag{3.9}$$

for all $g, h \in X$. Letting $x = x$,$y = z = 0$, f is even and $f(0) = 0$ in (3.3) we get

$$\|f(2x) + f(-x) + f(x) - f(x) - f(x) - f(x) - 3f(x)\| \le \phi(x,0,0)$$

$$\|f(2x) - 4f(x)\| \le \phi(x,0,0) \tag{3.10}$$

for all $x \in A$. So

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \le \frac{1}{4}\phi(x,0,0) \tag{3.11}$$

for all $x \in A$. Hence $d(f, Jf) \le \frac{1}{4}$. By Theorem 1.1, there exists a mapping $H : A \to B$ such that the following hold.

1. H is a fixed point of J, that is

$$H(2x) = 4H(x) \tag{3.12}$$

for all $x \in A$. The mapping H is a unique fixed point of J in the set

$$Y = \{g \in X : d(f,g) < \infty\} \tag{3.13}$$

This implies that H is a unique mapping satisfying (3.12) such that there exists $C \in (0, \infty)$ satisfying

$$\|H(x) - f(x)\|_B \le C\phi(x,0,0) \tag{3.14}$$

for all $x \in A$

2. $d(J^n f, H) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{f(2^n x)}{4^n} = H(x) \tag{3.15}$$

for all $x \in A$.

3. $d(f, H) \le \left(\frac{1}{1-L}\right) d(f, Jf)$ which implies the inequality

$$d(f,H) \le \frac{1}{4-4L} \quad . \tag{3.16}$$

This implies that the inequality (3.5) holds.

It follows from (3.2), (3.3) and (3.15) that

$$\begin{aligned}
&\|H(2x-y) + H(2y-z) + H(2z-x) + H(x+y+z) - H(x-y+z)\\
&- H(x+y-z) - H(x-y-z) - 3H(x) - 3H(y) - 3H(z)\|\\
&= \lim_{n\to\infty} \frac{1}{4^n}\|f(2^n(2x-y)) + f(2^n(2y-z)) + f(2^n(2z-x)) + f(2^n(x+y+z))\\
&- f(2^n(x-y+z)) - f(2^n(x+y-z)) - f(2^n(x-y-z))\\
&- 3f(2^n x) - 3f(2^n y) - 3f(2^n z)\|\\
&\le \lim_{n\to\infty} \frac{1}{4^n}\phi(2^n x, 2^n y, 2^n z) = 0
\end{aligned} \tag{3.17}$$

for all $x, y, z \in A$. So

$$\begin{aligned} &H(2x-y)+H(2y-z)+H(2z-x)+H(x+y+z)-H(x-y+z)\\ &-H(x+y-z)-H(x-y-z)=3H(x)+3H(y)+3H(z) \end{aligned} \tag{3.18}$$

for all $x, y, z \in A$. By Theorem 2.1, the mapping $H : A \to B$ is a quadratic.

By the same reasoning as in the proof of theorem of [4], the mapping $H : A \to B$ is $\Re$-linear.

It follows from (3.4) that

$$\begin{aligned} \|H(xy)-H(x)H(y)\|_B &= \lim_{n\to\infty}\frac{1}{16^n}\|f(4^n xy)-f(2^n x)f(2^n y)\|\\ &\le \lim_{n\to\infty}\frac{1}{16^n}\phi(2^n x,2^n y,0)\\ &\le \lim_{n\to\infty}\frac{1}{4^n}\phi(2^n x,2^n y,0)=0 \end{aligned} \tag{3.19}$$

for all $x, y \in A$. So

$$H(xy)=H(x)H(y) \tag{3.20}$$

for all $x, y \in A$. Thus $H : A \to B$ is a homomorphism satisfying (3.5) as desired. □

Corollary 3.2. *Let $r < 2$ and θ be non negative real numbers, and let $f : A \to B$ be a mapping such that*

$$\begin{aligned} \|Cf(x,y,z)\|_B &\le \theta\left(\|x\|_A^r+\|y\|_A^r+\|z\|_A^r\right)\\ \|f(xy)-f(x)f(y)\|_B &\le \theta(\|x\|_A^r+\|y\|_A^r) \end{aligned} \tag{3.21}$$

for all $x, y, z \in A$. If $f(tx)$ is continuous in $t \in \Re$ for each fixed $x \in A$, then there exists a unique homomorphism $H : A \to B$ such that

$$\|f(x)-H(x)\|_B \le \frac{\theta}{4-2^r}\|x\|_A^r \tag{3.22}$$

for all $x \in A$.

Proof. The proof follows from Theorem 3.1, by taking

$$\phi(x,y,z) := \theta\left(\|x\|_A^r+\|y\|_A^r+\|z\|_A^r\right) \tag{3.23}$$

for all $x, y, z \in A$, then we can choose $L = 2^{r-2}$ and we get the desired result.

$$\begin{aligned} \|f(x)-H(x)\|_B &\le \frac{1}{4-4L}\phi(x,0,0)\\ &\le \frac{1}{4-4(2^{r-2})}\phi(x,0,0)\\ &\le \frac{\theta}{4-2^r}\|x\|_A^r. \end{aligned}$$

□

Theorem 3.3. *Let $f : A \to B$ be a mapping for which there exists a function $\phi : A^3 \to [0,\infty)$ satisfying (3.3) and (3.4) such that*

$$\sum_{j=0}^{\infty}16^j\phi\left(\frac{x}{2^j},\frac{y}{2^j},\frac{z}{2^j}\right)<\infty \tag{3.24}$$

for all $x, y, z \in A$. If there exists an $L < 1$ such that $\phi(x, 0, 0) \leq \frac{1}{4}L\phi(2x, 0, 0)$ for all $x \in A$ and if f(tx) is continuous in $t \in \Re$ for each fixed $x \in A$, then there exists a unique homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{L}{4 - 4L}\phi(x, 0, 0) \tag{3.25}$$

for all $x \in A$.

Proof. we consider the linear mapping $J : X \to X$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right) \tag{3.26}$$

for all $x \in A$. It follows from (3.10) that

$$\left\|f(x) - 4f(\frac{x}{2})\right\| \leq \phi\left(\frac{x}{2}, 0, 0\right) \leq \frac{L}{4}\phi(x, 0, 0) \tag{3.27}$$

for all $x \in A$. Hence $d(f, Jf) \leq \frac{L}{4}$. By Theorem 1.1, there exists a mapping $H : A \to B$ such that the following holds

1. H is a fixed point of J, that is
$$H(2x) = 4H(x) \tag{3.28}$$
for all $x \in A$. The mapping H is a unique fixed point of J in the set
$$Y = \{g \in X : d(f, g) < \infty\} \quad . \tag{3.29}$$
This implies that H is a unique mapping satisfying (3.28) such that there exists $C \in (0, \infty)$ satisfying
$$\|H(x) - f(x)\|_B \leq C\phi(x, 0, 0) \tag{3.30}$$
for all $x \in A$
2. $d(J^n f, H) \to 0$ as $n \to \infty$. this implies the equality
$$\lim_{n\to\infty} 4^n f(\frac{x}{2^n}) = H(x) \tag{3.31}$$
for all $x \in A$.
3. $d(f, H) \leq \left(\frac{1}{1-L}\right) d(f, Jf)$ which implies the inequality
$$d(f, H) \leq \frac{L}{4 - 4L} \tag{3.32}$$

This implies that the inequality (3.25) holds.

It follows from (3.3), (3.24) and (3.31) that

$$\begin{aligned}
&\|H(2x - y) + H(2y - z) + H(2z - x) + H(x + y + z) - H(x - y + z) \\
&- H(x + y - z) - H(x - y - z) - 3H(x) - 3H(y) - 3H(z)\|_B \\
&= \lim_{n\to\infty} 4^n \|f\left(\frac{2x - y}{2^n}\right) + f\left(\frac{2y - z}{2^n}\right) + f\left(\frac{2z - x}{2^n}\right) + f\left(\frac{x + y + z}{2^n}\right) - f\left(\frac{x - y + z}{2^n}\right) \\
&- f\left(\frac{x + y - z}{2^n}\right) - f\left(\frac{x - y - z}{2^n}\right) - 3f\left(\frac{x}{2^n}\right) - 3f\left(\frac{y}{2^n}\right) - 3f\left(\frac{z}{2^n}\right)\| \\
&\leq \lim_{n\to\infty} 4^n\phi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \\
&\leq \lim_{n\to\infty} 16^n\phi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0
\end{aligned} \tag{3.33}$$

for all $x, y, z \in A$. So

$$\begin{aligned} &H(2x-y)+H(2y-z)+H(2z-x)+H(x+y+z)-H(x-y+z)\\ &-H(x+y-z)-H(x-y-z)=3H(x)+3H(y)+3H(z) \end{aligned} \tag{3.34}$$

for all $x, y, z \in A$. By Theorem 2.1, the mapping $H : A \to B$ is a quadratic.

By the same reasoning as in the proof of theorem of [4], the mapping $H : A \to B$ is $\Re$-linear.

It follows from (3.4) that

$$\begin{aligned} \|H(xy)-H(x)H(y)\|_B &= \lim_{n\to\infty} 16^n \left\| f\left(\frac{xy}{4^n}\right) - f\left(\frac{x}{2^n}\right) f\left(\frac{y}{2^n}\right) \right\| \\ &\le \lim_{n\to\infty} 16^n \phi\left(\frac{x}{2^n}, \frac{y}{2^n}, 0\right) = 0 \end{aligned} \tag{3.35}$$

for all $x, y \in A$. So

$$H(xy) = H(x)H(y) \tag{3.36}$$

for all $x, y \in A$. Thus $H : A \to B$ is a homomorphism satisfying (3.25) as desired. □

Corollary 3.4. *Let $r > 2$ and θ be non negative real numbers, and let $f : A \to B$ be a mapping satisfying (3.21). If f(tx) is continuous in $t \in \Re$ for each fixed $x \in A$, then there exists a unique homomorphism $H : A \to B$ such that*

$$\|f(x) - H(x)\|_B \le \frac{\theta}{2^r - 4} \|x\|_A^r \tag{3.37}$$

for all $x \in A$.

Proof. The proof follows from Theorem 3.3, by taking

$$\phi(x, y, z) := \theta\left(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r\right) \tag{3.38}$$

for all $x, y, z \in A$, then we can choose $L = 2^{2-r}$ and we get the desired result.

$$\begin{aligned} \|f(x) - H(x)\|_B &\le \frac{L}{4-4L}\phi(x, 0, 0) \\ &\le \frac{2^{2-r}}{4-4(2^{2-r})}\theta\left(\|x\|_A^r\right) \\ &\le \frac{\theta}{2^r-4}\left(\|x\|_A^r\right). \end{aligned}$$

□

4 Hyers-Ulam Stability of a Functional Equation

The following Corollaries provides the Hyers-Ulam stability [36] for the quadratic functional equation (1.1). This kind of stability considers both the sum and the product of powers of norms as an upper bound for the norm of the Cauchy difference.

Corollary 4.1. *Let $r < \frac{2}{3}$ and θ be non negative real numbers, and let $f : A \to B$ be a mapping such that*

$$\begin{aligned} \|Cf(x,y,z)\|_B &\le \theta\left(\|x\|_A^{3r} + \|y\|_A^{3r} + \|z\|_A^{3r} + \|x\|_A^r \|y\|_A^r \|z\|_A^r\right) \\ \|f(xy) - f(x)f(y)\|_B &\le \theta(\|x\|_A^{3r} + \|y\|_A^{3r} + \|x\|_A^r \|y\|_A^r) \end{aligned} \tag{4.1}$$

for all $x, y, z \in A$. If $f(tx)$ is continuous in $t \in \Re$ for each fixed $x \in A$, then there exists a unique homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{\theta}{2^2 - 2^{3r}} \|x\|_A^{3r} \tag{4.2}$$

for all $x \in A$.

Proof. The proof follows from Theorem 3.1, by taking

$$\phi(x, y, z) := \theta \left(\|x\|_A^{3r} + \|y\|_A^{3r} + \|z\|_A^{3r} + \|x\|_A^r \|y\|_A^r \|z\|_A^r \right) \tag{4.3}$$

for all $x, y, z \in A$, then we can choose $L = 2^{3r-2}$ and we get the desired result.

$$\begin{aligned} \|f(x) - H(x)\|_B &\leq \frac{1}{4 - 4L} \phi(x, 0, 0) \\ &\leq \frac{1}{4 - 4(2^{3r-2})} \phi(x, 0, 0) \\ &\leq \frac{\theta}{2^2 - 2^{3r}} \|x\|_A^{3r} . \end{aligned}$$

□

Corollary 4.2. *Let $r > \frac{2}{3}$ and θ be non negative real numbers, and let $f : A \to B$ be a mapping satisfying (4.1). If f(tx) is continuous in $t \in \Re$ for each fixed $x \in A$, then there exists a unique homomorphism $H : A \to B$ such that*

$$\|f(x) - H(x)\|_B \leq \frac{\theta}{2^{3r} - 2^2} \|x\|_A^{3r} \tag{4.4}$$

for all $x \in A$.

Proof. The proof follows from Theorem 3.3, by taking

$$\phi(x, y, z) := \theta \left(\|x\|_A^{3r} + \|y\|_A^{3r} + \|z\|_A^{3r} + \|x\|_A^r \|y\|_A^r \|z\|_A^r \right) \tag{4.5}$$

for all $x, y, z \in A$, then we can choose $L = 2^{2-3r}$ and we get the desired result.

$$\begin{aligned} \|f(x) - H(x)\|_B &\leq \frac{L}{4 - 4L} \phi(x, 0, 0) \\ &\leq \frac{2^{2-3r}}{4 - 4(2^{2-3r})} \theta \left(\|x\|_A^{3r} \right) \\ &\leq \frac{\theta}{2^{3r} - 2^2} \left(\|x\|_A^{3r} \right) . \end{aligned}$$

□

5 Stability of Generalized Derivations on Real Banach Algebras

Throughout this section, assume that A is a real Banach algebra with norm $\|\cdot\|_A$. For a given mapping $f : A \to A$, we define

$$\begin{aligned} Df(x, y, z) := {} & f(2x - y) + f(2y - z) + f(2z - x) + f(x + y + z) - f(x - y + z) \\ & - f(x + y - z) - f(x - y - z) - 3f(x) - 3f(y) - 3f(z) \end{aligned} \tag{5.1}$$

for all $x, y, z \in A$. Note that a $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a derivation if $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in A$

Definition 5.1. A generalized derivation $f : A \to A$ is $\Re$- linear and fulfills the generalized Leibniz rule

$$f(xyz) = f(xy)z - xf(y)z + xf(yz) \tag{5.2}$$

We prove the Hyers-Ulam-Rassias stability of generalized derivations on real Banach algebras [37, 38] for the functional equation $Df(x, y, z) = 0$.

Theorem 5.1. *Let $f : A \to A$ be a mapping for which there exists a function $\phi : A^3 \to [0, \infty)$ satisfying (3.2) such that*

$$\|Df(x, y, z)\|_A \leq \phi(x, y, z) \tag{5.3}$$

$$\|f(xyz) - f(xy)z + xf(y)z - xf(yz)\|_A \leq \phi(x, y, z) \tag{5.4}$$

for all $x, y, z \in A$. If there exists an $L < 1$ such that $\phi(x, 0, 0) \leq 4L\phi\left(\frac{x}{2}, 0, 0\right)$ for all $x \in A$ and if $f(tx)$ is continuous in $t \in \Re$ for each fixed $x \in A$, then there exists a unique generalized derivation $\delta : A \to A$ such that

$$\|f(x) - \delta(x)\|_A \leq \frac{1}{4 - 4L}\phi(x, 0, 0) \tag{5.5}$$

for all $x \in A$.

Proof. Consider the set

$$X := \{g : A \to A\} \tag{5.6}$$

and introduce the generalized metric on X:

$$d(g, h) = inf\left\{C \in \Re_+ : \|g(x) - h(x)\|_A \leq C\phi(x, 0, 0), \forall x \in A\right\}. \tag{5.7}$$

It is easy to show that (X, d) is complete. Now, we consider the linear mapping $J : X \to X$ such that

$$Jg(x) := \frac{1}{4}g(2x) \tag{5.8}$$

for all $x \in A$. By [[12],Theorem 3.1]

$$d(Jg, Jh) \leq Ld(g, h) \tag{5.9}$$

for all $g, h \in X$.

Letting $y = z = 0$,f is even and $f(0) = 0$ in (5.3) we get

$$\|f(2x) - 4f(x)\| \leq \phi(x, 0, 0) \tag{5.10}$$

for all $x \in A$. So

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \leq \frac{1}{4}\phi(x, 0, 0) \tag{5.11}$$

for all $x \in A$. Hence $d(f, Jf) \leq \frac{1}{4}$. By Theorem 1.1, there exists a mapping $\delta : A \to A$ such that the following hold.

1. δ is a fixed point of J, that is

   $$\delta(2x) = 4\delta(x) \tag{5.12}$$

   for all $x \in A$. The mapping δ is a unique fixed point of J in the set

   $$Y = \{g \in X : d(f, g) < \infty\} \quad . \tag{5.13}$$

   This implies that δ is a unique mapping satisfying (5.12) such that there exists $C \in (0, \infty)$ satisfying

   $$\|\delta(x) - f(x)\|_A \leq C\phi(x, 0, 0) \tag{5.14}$$

   for all $x \in A$

2. $d(J^n f, \delta) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{f(2^n x)}{4^n} = \delta(x) \tag{5.15}$$

for all $x \in A$.

3. $d(f, \delta) \leq \left(\frac{1}{1-L}\right) d(f, Jf)$ which implies the inequality

$$d(f, \delta) \leq \frac{1}{4 - 4L} \quad . \tag{5.16}$$

This implies that the inequality (5.5) holds.

It follows from (3.2), (5.3) and (5.15) that

$$\begin{aligned}
&\|\delta(2x-y) + \delta(2y-z) + \delta(2z-x) + \delta(x+y+z) - \delta(x-y+z) \\
&- \delta(x+y-z) - \delta(x-y-z) - 3\delta(x) - 3\delta(y) - 3\delta(z)\|_A \\
&= \lim_{n\to\infty} \frac{1}{4^n} \|f(2^n(2x-y)) + f(2^n(2y-z)) + f(2^n(2z-x)) + f(2^n(x+y+z)) \\
&- f(2^n(x-y+z)) - f(2^n(x+y-z)) - f(2^n(x-y-z)) - 3f(2^n x) - 3f(2^n y) - 3f(2^n z)\|_A \\
&\leq \lim_{n\to\infty} \frac{1}{4^n} \phi(2^n x, 2^n y, 2^n z) = 0.
\end{aligned} \tag{5.17}$$

for all $x, y, z \in A$. So

$$\begin{aligned}
&\delta(2x-y) + \delta(2y-z) + \delta(2z-x) + \delta(x+y+z) - \delta(x-y+z) \\
&- \delta(x+y-z) - \delta(x-y-z) = 3\delta(x) + 3\delta(y) + 3\delta(z)
\end{aligned} \tag{5.18}$$

for all $x, y, z \in A$. By Theorem 2.1, the mapping $\delta : A \to A$ is a quadratic.

By the same reasoning as in the proof of theorem of [4], the mapping $\delta : A \to A$ is $\Re$-linear.

It follows from (5.4) that

$$\begin{aligned}
&\|\delta(xyz) - \delta(xy)z + x\delta(y)z - x\delta(yz)\|_A \\
&= \lim_{n\to\infty} \frac{1}{64^n} \|f(8^n xyz) - f(4^n xy)2^n z + 2^n x f(2^n y) 2^n z - 2^n x f(4^n yz)\| \\
&\leq \lim_{n\to\infty} \frac{1}{64^n} \phi(2^n x, 2^n y, 2^n z) \\
&\leq \lim_{n\to\infty} \frac{1}{4^n} \phi(2^n x, 2^n y, 2^n z) = 0
\end{aligned} \tag{5.19}$$

for all $x, y, z \in A$. So

$$\delta(xyz) = \delta(xy)z - x\delta(y)z + x\delta(yz) \tag{5.20}$$

for all $x, y, z \in A$. Thus $\delta : A \to A$ is a homomorphism satisfying (5.5) as desired. □

Theorem 5.2. *Let $f : A \to A$ be a mapping for which there exists a function $\phi : A^3 \to [0, \infty)$ satisfying (5.3) and (5.4) such that*

$$\sum_{j=0}^{\infty} 64^j \phi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) < \infty \tag{5.21}$$

for all $x, y, z \in A$. If there exists an $L < 1$ such that $\phi(x,0,0) \leq \frac{1}{4}L\phi(2x,0,0)$ for all $x \in A$ and if f(tx) is continuous in $t \in \Re$ for each fixed $x \in A$, then there exists a unique homomorphism $\delta : A \to A$ such that

$$\|f(x) - \delta(x)\|_A \leq \frac{L}{4-4L}\phi(x,0,0) \tag{5.22}$$

for all $x \in A$.

Proof. we consider the linear mapping $J : X \to X$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right) \tag{5.23}$$

for all $x \in A$. It follows from (5.10) that

$$\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| \leq \phi\left(\frac{x}{2},0,0\right) \leq \frac{L}{4}\phi(x,0,0) \tag{5.24}$$

for all $x \in A$. Hence $d(f, Jf) \leq \frac{L}{4}$. By Theorem 1.1, there exists a mapping $\delta : A \to A$ such that the following holds.

1. δ is a fixed point of J, that is
$$\delta(2x) = 4\delta(x) \tag{5.25}$$
for all $x \in A$. The mapping δ is a unique fixed point of J in the set
$$Y = \{g \in X : d(f,g) < \infty\}. \tag{5.26}$$
This implies that δ is a unique mapping satisfying (5.25) such that there exists $C \in (0,\infty)$ satisfying
$$\|\delta(x) - f(x)\|_A \leq C\phi(x,0,0) \tag{5.27}$$
for all $x \in A$
2. $d(J^n f, \delta) \to 0$ as $n \to \infty$. this implies the equality
$$\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) = \delta(x) \tag{5.28}$$
for all $x \in A$.
3. $d(f,\delta) \leq \left(\frac{1}{1-L}\right) d(f, Jf)$ which implies the inequality
$$d(f,\delta) \leq \frac{L}{4-4L}. \tag{5.29}$$

This implies that the inequality (5.22) holds.

It follows from (5.3), (5.21) and (5.28) that

$$\begin{aligned}
&\|\delta(2x-y) + \delta(2y-z) + \delta(2z-x) + \delta(x+y+z) - \delta(x-y+z) \\
&- \delta(x+y-z) - \delta(x-y-z) - 3\delta(x) - 3\delta(y) - 3\delta(z)\|_A \\
&= \lim_{n\to\infty} 4^n \|f\left(\frac{2x-y}{2^n}\right) + f\left(\frac{2y-z}{2^n}\right) + f\left(\frac{2z-x}{2^n}\right) + f\left(\frac{x+y+z}{2^n}\right) - f\left(\frac{x-y+z}{2^n}\right) \\
&- f\left(\frac{x+y-z}{2^n}\right) - f\left(\frac{x-y-z}{2^n}\right) - 3f\left(\frac{x}{2^n}\right) - 3f\left(\frac{y}{2^n}\right) - 3f\left(\frac{z}{2^n}\right)\| \\
&\leq \lim_{n\to\infty} 4^n \phi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \\
&\leq \lim_{n\to\infty} 64^n \phi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0
\end{aligned} \tag{5.30}$$

for all $x, y, z \in A$. So

$$\begin{aligned}&\delta(2x-y)+\delta(2y-z)+\delta(2z-x)+\delta(x+y+z)-\delta(x-y+z)\\&-\delta(x+y-z)-\delta(x-y-z)=3\delta(x)+3\delta(y)+3\delta(z)\end{aligned} \tag{5.31}$$

for all $x, y, z \in A$. By Theorem 2.1, the mapping $\delta : A \to A$ is a quadratic.

By the same reasoning as in the proof of theorem of [4], the mapping $\delta : A \to A$ is $\Re$-linear.
It follows from (5.4) that

$$\begin{aligned}&\|\delta(xyz)-\delta(xy)z+x\delta(y)z-x\delta(yz)\|_A\\&=\lim_{n\to\infty}64^n\left\|f\left(\frac{xyz}{8^n}\right)-f\left(\frac{xy}{4^n}\right)\frac{z}{2^n}+\frac{x}{2^n}f\left(\frac{y}{2^n}\right)\frac{z}{2^n}-\frac{x}{2^n}f\left(\frac{yz}{4^n}\right)\right\|_A\\&\leq\lim_{n\to\infty}64^n\phi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right)=0\end{aligned} \tag{5.32}$$

for all $x, y, z \in A$. So

$$\delta(xyz) = \delta(xy)z - x\delta(y)z + x\delta(yz) \tag{5.33}$$

for all $x, y, z \in A$. Thus $\delta : A \to A$ is a generalized derivation satisfying (5.25) as desired. □

6 Conclusion

We obtained the general solution of (1.1) and proved the Hyers-Ulam stability, Hyers-Ulam-Rassias stability of homomorphisms in real Banach algebras and generalized derivations on real Banach algebras of a new quadratic functional equation (1.1) with better upper bound results using fixed point method.

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Competing Interests

Authors have declared that no competing interests exist.

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